ON GENERAL DECAY RATES OF SOME VISCOELASTIC SYSTEMS

BY

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To my Family

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DISSERTATION ABSTRACT

NAME: JAMILU HASHIM HASSAN

TITLE OF STUDY: ON GENERAL DECAY RATES OF SOME VISCOELASTIC SYSTEMS

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In this dissertation, we investigate the asymptotic behavior of some viscoelastic systems, namely; viscoelastic-type Timoshenko and Bresse sytems, and a system of viscoelastic wave equations. We study viscoelastic-type Bresse and Timoshenko systems with relaxation function satisfying, for some constant $1 \leq p < 2$,

$$g'(t) \leq -\xi(t)g^p(t), \qquad \forall\, t \geq 0,$$

where $\xi : \mathbb{R}_+ \longrightarrow (0, +\infty)$ *is a non-increasing differentiable function. We prove some general decay results for the energy associated to solution of each system in the case of equal and non-equal speeds of wave propagation. For the system of viscoelastic wave equations, we consider relaxation functions satisfying, for some non-negative functions*

ξ_i *and* H_i,

$$g_i'(t) \leq -\xi_i(t)H_i(g_i(t)), \qquad \forall\, t \geq 0, \qquad \text{and} \qquad i = 1, 2,$$

and prove a new general decay result for the energy associated to solution of the system. Our results improve and generalize most of the existing results in the literature related to above systems and allow a wider class of relaxation functions.

ملخص الرسالة

الاسم: جميل هاشم حسن

عنوان الدراسة: حول معدّل الإضمحلال العام لبعض أنظمة المرونة اللزجة

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ملخص الرسالة في هذه الرسالة ، نقوم ببحث السلوك التقاربي لبعض أنظمة اللزوجة المرنة من نوع تيموشينكو وبريس، ونظام لمعادلات أمواج. ندرس نظامي تيموشينكو وبريس بدالة استرخاء تحقق:

$$g'(t) \leq -\xi(t)g^p(t), \qquad \forall, t \geq 0, \quad 1 \leq p < 2,$$

$\xi : \mathbb{R} \longrightarrow (0, \infty)$ دالة قابلة للاشتقاق وتنافصية. فنثبت بعض نتائج الاضمحلال لطاقة الحلول لهذه الأنظمة في حالة تساوي سرعات انتشار الأمواج وفي حالة عدم تساويها. أما بالنسبة لنظام معادلات الأمواج، فنعتبردوال استرخاء تحقق، لبعض الدوال الموجبة ξ_i و H_i ،

$$g_i'(t) \leq -\xi(t)H_i(g(t)), \qquad \forall\, t \geq 0, \quad i = 1, 2$$

ونثبت بعض نتائج الإضمحلال الجديدة لطاقة الحلول لهذه الأنظمة. فنتائجنا تعمّم وتحسّن جميع النتائج الموجودة ذات الصلة بموضوع دراسنا وتسمح باستعمال مجموعة أكبر من دوال الإسترخاء.

CHAPTER 1

INTRODUCTION

In continuum mechanics, elastic materials and viscous fluids are mostly considered. An elastic material is a material in which at each material point the stress at the present time depends completely on the current value of the strain. For an incompressible viscous fluid, the stress at any given point depends on the value of the velocity gradient at that point. When a material exhibits both elastic and viscous behaviors it is called viscoelastic material. Precisely, for viscoelastic materials the stress at any given point depends on the present values of strain and velocity gradient. Examples of viscoelastic materials include, but not limited to, human tissue, disk in the human spine, wood, compressible gas, metals at very high temperature, concrete, plastic and polymeric materials. Some viscoelastic materials such as polymers, suspensions and emulsions can not be described in this way. For such materials, the stress at any given point does not depend only on the values of strain and velocity gradient at that point, but also on the entire history of the motion, that is, they posses a memory effect. Therefore, this type of viscoelastic behavior is modeled by equation with memory; the differential equation which is influenced by the past values of some variables under consideration.

Amongst the early contributors in this field are: Boltzmann, Kelvin, Maxwell and Voigt.

Consider a bar of uniform cross-section which occupies the unit interval $(0,1) \subset \mathbb{R}$ in unstressed state. A typical particle in $(0,1)$ is denoted by x, to describe the evolution of particles in $(0,1)$, we let $u(x,t)$ represents the displacement of the particle at time t and reference position x. The strain ϵ is given by

$$\epsilon(x,t) := u_x(x,t), \tag{1.1}$$

and the balance of linear momentum takes the form

$$u_{tt}(x,t) = \sigma_x(x,t) + f(x,t), \qquad x \in (0,1), \quad t > 0, \tag{1.2}$$

where σ is the stress and f is an external force per unit mass. In 1874, Boltzmann [1] proposed that for material with memory the constitutive relation for small deformation is given by

$$\sigma(x,t) = \beta\epsilon(x,t) + \int_{-\infty}^{t} g(t-s)\big(\epsilon(x,t) - \epsilon(x,s)\big)ds, \tag{1.3}$$

where β is a non-negative constant and g is positive non-increasing function defined on $[0,\infty)$. In the case where $g \in L^1(0,\infty)$, equation (1.3) takes the form

$$\sigma(x,t) = c^2\epsilon(x,t) - \int_{-\infty}^{t} g(t-s)\epsilon(x,s)ds, \tag{1.4}$$

where $c^2 := \beta + \displaystyle\int_0^\infty g(s)ds$ measures the instantaneous response of stress to strain.

A substitution of (1.4) into (1.2) yields

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) + \int_{-\infty}^{t} g(t-s) u_{xx}(x,s) ds = f(x,t), \qquad x \in (0,1), \quad t > 0. \quad (1.5)$$

The function u is assumed to be known for any $t \leq 0$, that is, we have the following initial data:

$$u(x,t) = u_0(x,-t), \qquad u_t(x,0) = u_1(x) \qquad \forall\, x \in (0,1),\ \forall\, t \leq 0, \qquad (1.6)$$

we further assume that $f \equiv 0$. In order to study system (1.5)–(1.6), Dafermos (see [2], [3]) introduced a history function of the form

$$\eta^t(s) := u(t) - u(t-s) \qquad \forall\, t, s > 0.$$

This allowed him to write problem (1.5)–(1.6) in the form of first-order evolution equation and took advantage of some powerful tools in the theory of dynamical systems. For more details on the theory of viscoelasticity, see [1], [4], [5].

1.1 Literature Review

In this section we will give some literature review concerning viscoelastic wave equation, Timoshenko system, Bresse system and system of viscoelastic wave equations.

1.1.1 Viscoelastic Equation

For almost a half century, the asymptotic behavior for viscoelastic systems had been extensively studied by many researchers since the pioneer work of Dafermos [2], [6] in which he investigated a one-dimensional viscoelastic equation and proved the well-posedness of the problem provided that the relaxation function is positive and integrable. He also established that the solution decays asymptotically to zero if, in addition, the relaxation function is a monotone non-increasing smooth function. However, the rate of decay of the solution was not explicitly given. Hrusa [7] in 1985, considered the following one-dimensional viscoelastic problem with nonlinearity in the memory term:

$$
\begin{cases}
u_{tt} - cu_{xx} + \displaystyle\int_0^t g(t-s)\big(\psi(u_x(x,s))\big)_x ds = f(x,t), & \text{in} \quad (0,1)\times(0,\infty), \\
u(0,t) = u(1,t) = 0, & t \geq 0, \\
u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in [0,1],
\end{cases}
\tag{1.7}
$$

where $c > 0$ is a constant u_0, u_1 are given initial data and ψ is a nonlinear function. Under certain conditions on ψ, he established the global existence of a strong solution to problem (1.7) and showed that the solution decays exponentially to zero, if the relaxation function g decays exponentially to zero.

For multi-dimensional viscoelastic problems, we start with the work of Dassios and Zafiropoulos [8] in 1990, in which the authors studied a three-dimensional viscoelastic problem in the whole space $\mathbb{R}^3$ and proved a polynomial decay result for an exponentially decaying relaxation function. In 1994, Rivera [9] established an exponential decay result for sum of the first and second energies of a linear viscoelastic problem in a bounded domain of $\mathbb{R}^n$ with an exponentially decaying relaxation function by imposing some extra conditions on the second derivative of the relaxation function. Rivera and Lapa [10] improved this later result by proving a polynomial decay rate of the problem with a relaxation function that decays polynomially. In 2002, Cavalcanti *et al*. [11] studied the following problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(s)ds + \gamma(x)u_t = 0, & \text{in } \ \Omega \times (0,\infty), \\ u = 0, & \text{on } \ \partial\Omega \times [0,\infty), \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, & \text{in } \ \Omega, \end{cases} \tag{1.8}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$ and $\gamma : \Omega \longrightarrow \mathbb{R}_+$ is bounded and satisfies

$$\gamma(x) \geq \gamma_0 \quad a.e. \quad \text{on} \quad \omega \subset \Omega.$$

They imposed the following assumptions on the relaxation function, g :

$$g(0) > 0, \qquad \int_0^\infty g(s)ds < 1,$$

and there exist two positive constants ξ_1, ξ_2 such that

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \qquad \forall\, t \geq 0. \tag{1.9}$$

They proved an exponential rate of decay for the solution of (1.8) under some geometric conditions on ω. Berrimi and Messaoudi [12] showed that one can drop the geometric condition imposed on ω in [11] and still maintained the exponential decay of the solution of (1.8). They established their result under weaker conditions on g. Furthermore, the same authors in [13] extended and improved their result to the case where a source term is competing with a viscoelastic damping.

Up to the year 2008, most of the studies of viscoelastic problems were concerned with relaxation functions satisfying

$$g'(t) \leq -\xi g^p(t), \qquad \forall\, t \geq 0, \tag{1.10}$$

where $\xi > 0$ and $1 \leq p < \frac{3}{2}$ which, in turn, yielded either uniform or polynomial decay. In 2008, Messaoudi [14], [15] proved a more general decay rate from which the exponential and polynomial decay rates are only special cases. Precisely, he studied the following problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(s)ds = \gamma |u|^{m-2}u, & \text{in } \ \Omega \times (0,\infty), \\ u = 0, & \text{on } \ \partial\Omega \times [0,\infty), \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, & \text{in } \ \Omega, \end{cases} \tag{1.11}$$

with $\gamma = 0$ or $\gamma = 1$ and g satisfying

$$g'(t) \leq -\xi(t) g(t), \qquad \forall\, t \geq 0, \tag{1.12}$$

where $\xi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a non-increasing differentiable function; and showed that the energy of the solution of (1.11) decays with the same rate as g. Motivated by these results of Messaoudi, many general decay results using (1.12) have been established. See Han and Wang [16], Liu [17], [18], Cao [19] and references therein.

In 2009, Alabau-Boussouira and Cannarsa [20] announced, without a proof, a general decay result for the solution of problem (1.11) with $\gamma = 0$ for a class of relaxation functions satisfying

$$g'(t) \le -H(g(t)), \qquad \forall\, t \ge 0,$$

where $H : [0, \infty] \longrightarrow [0, \infty]$ is a strictly increasing, strictly convex and C^1 function on $[0, k_0]$ with $H(0) = H'(0) = 0$ and satisfies the following extra conditions:

$$\int_0^{k_0} \frac{1}{H(s)} ds = \infty, \qquad \int_0^{k_0} \frac{s}{H(s)} ds < 1 \qquad \text{and} \qquad \liminf_{s \to 0^+} \frac{H(s)}{sH(s)} > \frac{1}{2}.$$

Moreover, if H satisfies

$$g'(t) = -H(g(t)), \qquad \forall\, t \ge 0 \qquad \text{and} \qquad \limsup_{s \to 0^+} \frac{H(s)}{sH'(s)} < 1,$$

then an explicit optimal decay rate is claimed. They also asked the following question:

Q. What about a more general class of relaxation functions satisfying

$$g'(t) \le -\xi(t) H(g(t)), \qquad \forall\, t \ge 0?$$

It is worth noting that the result of Messaoudi in [14] answered **Q** when $H = Id$ and ξ is a positive nonincreasing differentiable function. In 2012, Mustafa and Messaoudi [21] relaxed most of the unnecessary conditions imposed on H in [20] and answered **Q** with $\xi \equiv 1$. In 2016, Messaoudi and Al-Khulaifi [22] proved a general and optimal decay rate of the solution of (1.11) with $\gamma = 0$ for a class of relaxation functions, satisfying

$$g'(t) \leq -\xi(t)g^p(t), \qquad \forall\, t \geq 0, \qquad 1 \leq p < \frac{3}{2}. \tag{1.13}$$

This result answered **Q** with ξ being a nonincreasing differentiable function and $H(s) = s^p$, for $1 \leq p < \frac{3}{2}$. Very recently, Mustafa [23] gave a complete answer to **Q** by assuming that H is either linear or strictly increasing and strictly convex C^2 function on $(0, r]$, for $r \leq g(0)$ and ξ is a positive nonicreasing differentiable function. His result generalizes and improves all the existing results in the literature related to the decay of the solution of viscoelastic equations.

1.1.2 Timoshenko System

In 1921, Timoshenko [24] presented the following system of hyperbolic partial differential equations

$$\begin{aligned} &\rho u_{tt} = (K(u_x - \phi))_x && \text{in } (0, L) \times (0, +\infty), \\ &I_\rho \phi_{tt} = (EI\phi_x)_x + K(u_x - \phi) && \text{in } (0, L) \times (0, +\infty), \end{aligned} \tag{1.14}$$

as a mathematical model describing the dynamics of a beam by taking the transverse shear strain into consideration. Here t represents the time and x is the space variable along the beam of length L, u is the transverse displacement of the beam from its equilibrium configuration and ϕ is the rotational angle of the filament of the beam. The constant coefficients $\rho,\ I_\rho,\ E,\ I$ and K are the mass density, the polar moment of inertia of a cross-section, the Young modulus of elasticity, the moment of inertia of a cross-section, and the shear modulus, respectively.

For almost a century, a great number of researchers have devoted a considerable amount of time studying this model and many results concerning the well-posedness and long-time behaviour of the system have been discussed. Various types of dissipation mechanisms (such as boundary and/or internal controls) were employed in order to achieve different stability results. Let us mention a few of these results from the literature. For more details we refer the reader to the references in this dissertation and the references therein.

In the case of boundary feedback controls, Kim and Renardy [25] investigated the uniform stabilization of (1.14) with clamped end at $x = 0$, that is,

$$u(0,t) = 0, \quad \phi(0,t) = 0, \qquad \forall t \geq 0$$

and a mixed boundary conditions of the form

$$\begin{aligned} &K\phi(L,t) - Ku_x(L,t) = \alpha u_t(L,t), && \forall t \geq 0 \\ &EI\phi(L,t) = -\beta\phi_t(L,t), && \forall t \geq 0. \end{aligned}$$

They used the multiplier method to prove that the energy associated to system (1.14) decays exponentially. Feng *et al*. [26] considered the problem in [25] but replaced the linear boundary controls with some nonlinear feedback controls and established the asymptotic and exponential stability of the system by using LaSalle invariance principle and energy perturbation method. Messaoudi and Mustafa in [14] investigated the long-time behavior of a Timoshenko system with internal and/or boundary feedback controls. Without imposing any restrictive growth assumption on the damping terms near the origin, they established explicit and general decay results.

In the presence of two internal feedback controls, Raposo *et al.* [27] established the exponential decay of the solution of a linear Timoshenko-type beam equation with linear frictional dissipative terms. Precisely, they studied the following system

$$\begin{cases} \rho_1 u_{tt} - K(u_x - \psi)_x + u_t = 0, & 0 < x < L,\ t > 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(u_x - \psi) + \psi_t = 0, & 0 < x < L,\ t > 0, \\ u(0,t) = u(L,t) = \psi(0,t) = \psi(L,t) = 0, & \forall t > 0 \end{cases} \tag{1.15}$$

and used the semigroup method developed by Liu and Zheng [28] and proved the exponential decay of the solution of the system (1.15).

However, when a control is present on the rotation angle or on the transverse displacement alone the decay rates turn out to depend on the constants $\rho,\ I_\rho,\ E,\ I$ and K. For instance, Soufyane and Wehbe [29] proved that one can uniformly stabilize a linear Timoshenko system under influence of one locally distributed damping. Indeed, they considered the following problem

$$\begin{cases} \rho_1 u_{tt} = (K(u_x - \psi))_x, & 0 < x < L,\ t > 0, \\ \rho_2 \psi_{tt} = (b\psi_x)_x + K(u_x - \psi) - \sigma \psi_t, & 0 < x < L,\ t > 0, \\ u(0,t) = u(L,t) = \psi(0,t) = \psi(L,t) = 0, & \forall t > 0, \end{cases} \tag{1.16}$$

where σ is a continuous function on $[0, L]$ satisfying

$$\sigma(x) \geq \gamma_0 > 0, \quad \forall x \in [c, d] \subset [0, L].$$

They proved the exponential stability for the system (1.16) if and only if the system has equal speeds of wave propagation, that is, if and only if

$$\frac{\rho_1}{K} = \frac{\rho_2}{b} \tag{1.17}$$

holds. Otherwise, only the asymptotic stability is established. Fernandez Sare and Rivera [30] studied a Timoshenko system with infinite history of the form

$$\begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0 & \text{in} \ \ (0, L) \times (0, +\infty), \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \displaystyle\int_0^{+\infty} g(s)\psi_{xx}(t-s)ds = 0 & \text{in} \ \ (0, L) \times (0, +\infty), \\ \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, \end{cases} \tag{1.18}$$

where the relaxation function g satisfies

$$g(t) > 0,\ \exists k_0, k_1, k_2 > 0 : -k_0 g(t) \leq g'(t) \leq -k_1 g(t),\ |g''(t)| \leq k_2 g(t) \tag{1.19}$$

and

$$\widetilde{b} := b - \int_0^{+\infty} g(s)ds > 0. \tag{1.20}$$

They showed that the system (1.18) is exponentially stable if and only if relation (1.17) holds, otherwise it is polynomially stable. Messaoudi and Said-Houari [31] investigated the same system with the following conditions on g:

$$g(t) > 0, \quad \exists k_0 > 0 : g'(t) \leq -k_0 g^p(t),\ 1 \leq p < \frac{3}{2}, \quad \widetilde{b} := b - \int_0^{+\infty} g(s)ds > 0 \tag{1.21}$$

and proved that if (1.17) holds, then the energy associated to the system decays exponentially for $p = 1$ and polynomially for $p > 1$. However, if (1.17) is not satisfied, they established the decay rate of the type $\frac{1}{t^{1/(2p-1)}}$. Their result generalizes and improves that of [30]. In [32], Guesmia *et al.* looked into (1.18) with a relaxation function g having more general decay, and obtained some general decay results, from which the exponential and polynomial decay results are only special cases. Additionally, they improved the result of [30] and [31].

The stability of a linear viscoelastic-type Timoshenko system (finite history) has also attracted considerable attention of researchers. For example, Ammar-Khodja *et al.* [33] studied the following system

$$
\begin{aligned}
&\rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0 && \text{in } (0,L)\times(0,+\infty),\\
&\rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x+\psi) + \int_0^t g(t-s)\psi_{xx}(s)ds = 0 && \text{in } (0,L)\times(0,+\infty),
\end{aligned}
\tag{1.22}
$$

with Dirichlet-boundary conditions. They proved that this system decays uniformly if and only if the coefficients satisfy (1.17). Concerning the rate of decay, they showed that if g satisfies hypotheses (1.19) and (1.20), then the system is exponentially stable. If g is of polynomial type, that is, if it satisfies, for some positive constants b_0, b_1, b_2, b_3, b_4 and $p > 2$,

$$
\begin{aligned}
&0 < g(t) \le b_0(1+t)^{-p}, \qquad -b_1 g^{\frac{p+1}{p}}(t) \le g'(t) \le -b_2 g^{\frac{p+1}{p}}(t),\\
&-b_3|g'(t)|^{\frac{p+2}{p+1}} \le g''(t) \le -b_4|g'(t)|^{\frac{p+2}{p+1}},
\end{aligned}
$$

then the energy associated to the system decays polynomially to zero. In case of the coefficients of system (1.22) satisfying (1.17), Guesmia and Messaoudi [34] established the same stabilization results of [33] under conditions (1.21) which are weaker than the ones in [33]. Also, Messaoudi and Mustafa [35] discussed system (1.22) and proved a general decay result from which the exponential and polynomial stability are only special cases under the conditions

$$
g(t) > 0, \quad g'(t) \le -\xi(t)g(t), \quad b - \int_0^{+\infty} g(s)ds := l > 0,
$$

where ξ is positive non-increasing differentiable function. In fact, the result of [35] generalizes those of [33] and [34] and allows a wider class of relaxation functions. In

2013, Almeida Júnior *et al.* [36] considered the situation when the control is only on the transverse displacement equation, which is more realistic from the physical point of view. Precisely, they studied the following system

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + \mu\varphi_t = 0 \quad \text{in } (0,L)\times(0,+\infty) \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) = 0 \quad \text{in } (0,L)\times(0,+\infty) \end{cases}$$

and showed that the effect of linear frictional damping on the first equation stabilizes the system exponentially if (1.17) holds, otherwise, the stabilization is of polynomial type. This result was later improved and generalized by Guesmia and Messaoudi [37]. For more recent results on this and viscoelastic systems in general, see [22], [38], [39], [40], [41].

1.1.3 Bresse System

Bresse system is a mathematical model that describes the vibration of a planar, linear shearable curved beam. The model was first derived by Bresse [42] and it consists of three coupled wave equations given by

$$\begin{aligned} &\rho_1\varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - lk_3(w_x - \varphi) + F_1 = 0 && \text{in } (0,L)\times(0,\infty), \\ &\rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + lw) + F_2 = 0 && \text{in } (0,L)\times(0,\infty), \\ &\rho_1 w_{tt} - k_3(w_x - \varphi)_x + lk_1(\varphi_x + \psi + lw) + F_3 = 0 && \text{in } (0,L)\times(0,\infty), \end{aligned} \tag{1.23}$$

where φ, ψ, w represent the vertical displacement, the shear angle, and the longitudinal displacement, respectively; $\rho_1, \rho_2, k_1, k_2, k_3, l$ are positive parameters and F_1, F_2, F_3 are external forces.

A lot of results dealing with well-posedness and asymptotic behaviour of the above system have been published. We start with the work of Santos *et al*. [43] in 2010, where they studied the Bresse system with Dirichlet-Dirichlet-Dirichlet boundary conditions and linear frictional damping acting on each equation, that is,

$$(F_1, F_2, F_3) = (\gamma_1 \varphi_t,\ \gamma_2 \psi_t,\ \gamma_3 w_t), \tag{1.24}$$

where $\gamma_1, \gamma_2, \gamma_3 > 0$. They established an exponential decay rate for the system using spectral theory approach developed by Z. Liu and S. Zheng in [28]. They also gave a numerical scheme using finite difference method to illustrate their theoretical result. Soriano *et al*. [44] used the method developed by Lasiecka and Tataru in [45] and proved a uniform decay rate for the same system with a nonlinear frictional damping acting on the second equation and locally distributed nonlinear damping acting on the other equations. Precisely, the external forces are given by

$$(F_1, F_2, F_3) = (\alpha(x) g_1(\varphi_t),\ g_2(\psi_t),\ \gamma(x) g_3(w_t))$$

with $\alpha, \gamma \in L^\infty(0, L)$ and the g_i's are continuous and monotone increasing functions. The results of [43] and [44] were established without imposing any restriction on the speeds of wave propagation given by

$$s_1 = \sqrt{\frac{k_1}{\rho_1}}, \qquad s_2 = \sqrt{\frac{k_2}{\rho_2}}, \qquad \text{and} \qquad s_3 = \sqrt{\frac{k_3}{\rho_1}}. \tag{1.25}$$

Alves *et al*. [46] used the semigroup and spectral theory to obtain the exponential stability of the Bresse system with three controls at the boundary.

In the presence of dissipating terms in only one or two of the equations in system (1.23), the decay rates of the energy associated to the system depend totally on the speeds of the wave propagation. As illustrated in [47], Alabau-Boussouira *et al*. studied (1.23) with a linear frictional damping acting on the second equation; that is, they used (1.24), with $\gamma_1 = \gamma_3 = 0$ and $\gamma_2 > 0$ and showed that the system is exponentially stable if and only if it has equal speeds of wave propagation,

$$\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} = \frac{k_3}{\rho_3}. \tag{1.26}$$

As mentioned by many authors [47], [48], relation (1.26) is physically unrealistic. In the case of non-equal speeds of wave propagation, they proved polynomial stability with rates which can be improved with the regularity of the initial data. Fatori and Monteiro [49] improved this result in the case non-equal speeds of wave propagation by proving an optimal decay rate. Soriano *et al*. [50] established the same exponential stability result as in [47] by replacing the frictional damping with indefinite one; that is, they replaced γ_2 in [47] with a function $a : (0, L) \longrightarrow \mathbb{R}$ such that $\bar{a} = \dfrac{1}{L}\displaystyle\int_0^L a(x)dx >$ 0 and $\|a - \bar{a}\|_{L^2(0,L)}$ is small enough. Wehbe and Youcef [51] inspected the situation of two locally distributed dampings acting on the last two equations; that is,

$$(F_1, F_2, F_3) = (0,\, a_1(x)\psi_t,\, a_2(x)w_t),$$

where $a_i : (0, L) \longrightarrow \mathbb{R}$ are non-negative functions which can take value zero on some part of the interval $(0, L)$. By using the frequency domain and the multiplier methods, they showed that the system is exponentially stable if and only if $s_1 = s_2$. When $s_1 \neq s_2$ they established a polynomial decay rate which can be improved with the regularity of the initial data. The same result was established by Alves *et al.* in [48], in the case of non-equal speeds of wave propagation, by using a recent result of Borichev and Tomilov in [52] they showed that the solution is polynomially stable with optimal decay rate.

Concerning the dissipation via heat effect, we mention the work of Liu and Rao [53], where the following system

$$
\begin{array}{ll}
\rho_1\varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - lk_3(w_x - \varphi) + l\gamma\chi = 0 & \text{in } (0, L) \times (0, \infty), \\
\rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + lw) + \gamma\theta_x = 0 & \text{in } (0, L) \times (0, \infty), \\
\rho_1 w_{tt} - k_3(w_x - \varphi)_x + lk_1(\varphi_x + \psi + lw) + \gamma\chi_t = 0 & \text{in } (0, L) \times (0, \infty), \\
\rho_3\theta_t - \theta_{xx} + \gamma\psi_{xt} = 0 & \text{in } (0, L) \times (0, \infty), \\
\rho_3\chi_t - \chi_{xx} + \gamma(w_x - l\varphi)_t = 0 & \text{in } (0, L) \times (0, \infty),
\end{array}
\tag{1.27}
$$

with boundary and initial conditions was considered. They showed that the exponential stability of the system is equivalent to the validity of the identity (1.26). In the case where (1.26) does not hold, they established a polynomial-type decay rate. Fatori and Muñoz Rivera [54] obtained a similar result as in [53] for the thermoelastic Bresse system (1.27) when the fifth equation is omitted. They also showed that the polynomial decay rate is optimal in the case of non-equal speeds of wave propagation. Filippo

Dell'Oro [55] gave a detail stability analysis of the thermoelastic Bresse-Gurtin-Pipkin system of the form:

$$
\begin{aligned}
&\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - lk_0(w_x - \varphi) = 0 \quad \text{in } \ (0,L)\times(0,\infty),\\
&\rho_2\psi_{tt} - k_2\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\theta_x = 0 \quad \text{in } \ (0,L)\times(0,\infty),\\
&\rho_1 w_{tt} - k_0(w_x - \varphi)_x + lk(\varphi_x + \psi + lw) = 0 \quad \text{in } \ (0,L)\times(0,\infty),\\
&\rho_3\theta_t - k_1\int_0^\infty g(s)\theta_{xx}(t-s)ds + \gamma\psi_{xt} = 0 \quad \text{in } \ (0,L)\times(0,\infty),
\end{aligned}
\tag{1.28}
$$

where g is a bounded convex integrable function on $[0,\infty)$ satisfying

$$
\int_0^\infty g(s)ds = 1,
$$

and there exists a non-increasing absolutely continuous function $\mu : (0,\infty) \longrightarrow [0,\infty)$ such that

$$
\mu(0) = \lim_{s\to 0}\mu(s) \in (0,\infty), \qquad g(s) = \int_s^\infty \mu(\tau)d\tau, \qquad \forall\, s\in[0,\infty)
$$

and

$$
\mu'(s) + \nu\mu(s) \le 0 \quad \text{for} \ \text{ some } \ \nu > 0 \quad \text{and} \quad a.e.\ s \in (0,\infty).
$$

By introducing a new stability number of the form

$$
\chi_g = \left(\frac{\rho_1}{\rho_3 k} - \frac{1}{g(0)k_1}\right)\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right) - \frac{1}{g(0)k}\frac{\rho_1\gamma^2}{\rho_3 bk},
$$

he proved that the semigroup generated by (1.28) is exponentially stable if and only if

$$\chi_g = 0 \qquad \text{and} \qquad k = k_0.$$

As a special case, he showed that his stability result gave the stability characterization of Bresse systems with Fourier, Maxwell-Cataneo and Coleman-Gurtin thermal dissipation. The reader is referred to [56], [57], [58], [59], [60], [61], [62], [63] and the references therein for more recent results on thermoelastic Bresse systems.

There are few results that dealt with stabilization of Bresse system via infinite memory. We begin with the work of Guesmia and Kafini [64] in 2015. They studied the following system

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - lk_3(w_x - l\varphi) + \displaystyle\int_0^\infty g_1(s)\varphi_{xx}(x, t-s)ds = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + lw) + \displaystyle\int_0^\infty g_2(s)\psi_{xx}(x, t-s)ds = 0, \\ \rho_1 w_{tt} - lk_3(w_x - l\varphi)_x + lk_1(\varphi_x + \psi + lw) + \displaystyle\int_0^\infty g_3(s)w_{xx}(x, t-s)ds = 0, \\ \varphi(0,t) = \psi(0,t) = w(0,t) = \varphi(L,t) = \psi(L,t) = w(L,t) = 0, \\ \varphi(x,-t) = \varphi_0(x,t),\ \ \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,-t) = \psi_0(x,t),\ \ \psi_t(x,0) = \psi_1(x), \\ w(x,-t) = w_0(x,t),\ \ w_t(x,0) = w_1(x), \end{cases} \tag{1.29}$$

where $(x,t) \in (0,L) \times \mathbb{R}_+,\ g_i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are differentiable non-increasing and integrable functions, and L, l_i, ρ_i, k_i are positive constants. They proved the well-posedness and the asymptotic stability of (1.29). Later, Guesmia and Kirane [65] used two infinite memories to obtain the same stability result of [64] under the following conditions on the speeds of wave propagation:

$$\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} \quad \text{in case } g_1 = 0, \qquad \frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} \quad \text{in case } g_2 = 0, \qquad \frac{k_1}{\rho_1} = \frac{k_3}{\rho_3} \quad \text{in case } g_3 = 0.$$

Santos *et al*. [66] discussed the Bresse system with only one infinite memory acting on the shear angle displacement equation. Precisely, they studied problem (1.29) with

$$g_1 = g_3 = 0 \qquad \text{and} \qquad g_2 \quad \text{satisfying}: \quad -\alpha_1 g_2(t) \leq g_2'(t) \leq -\alpha_2 g_2(t), \quad \forall\, t \geq 0,$$

for some $\alpha_1, \alpha_2 > 0$. They showed that the solution of the system decays exponentially to zero if and only if (1.26) holds, otherwise a polynomial stability of the system with an optimal decay rate of type $t^{-1/2}$ was obtained. Recently, Guesmia [67] analysed the asymptotic stability of Bresse system with one infinite memory in the longitudinal displacement.

To the best of our knowledge, there is no result in the literature that deals with the stability of Bresse system via viscoelastic damping of finite memory-type.

1.1.4 System of Viscoelastic Wave Equations

For the general decay results of solutions for system of viscoelastic wave equations, Messaoudi and Tatar [68] studied the following problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(s)ds + f(u,v) = 0, & \text{in } \ \Omega \times (0,\infty), \\ v_{tt} - \Delta v + \displaystyle\int_0^t h(t-s)\Delta v(s)ds + k(u,v) = 0, & \text{in } \ \Omega \times (0,\infty), \\ u = v = 0, & \text{on } \ \partial\Omega \times [0,\infty), \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, \quad v(\cdot,0) = v_0, \quad v_t(\cdot,0) = v_1, & \text{in } \ \Omega, \end{cases} \tag{1.30}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$ and u_0, v_0, u_1, v_1 are given initial data. The functions f and k satisfy, for all $(u,v) \in \mathbb{R}^2$, the following assumption:

$$\begin{cases} |f(u,v)| \leq d(|u|^{\beta_1} + |v|^{\beta_2}) \\ |k(u,v)| \leq d(|u|^{\beta_3} + |v|^{\beta_4}) \end{cases}$$

for some constant $d > 0$ and

$$\beta_i \geq 1 \qquad (n-2)\beta_i < n, \qquad i = 1,2,3,4.$$

Under the following hypothesis: there exist two positive constants ξ_1, ξ_2 such that

$$g'(t) \le -\xi_1 g^p(t), \quad t \ge 0, \quad 1 \le p < \tfrac{3}{2}$$
$$h'(t) \le -\xi_2 h^q(t), \quad t \ge 0, \quad 1 \le q < \tfrac{3}{2}$$

they proved an exponential decay result if $(p, q) = (1, 1)$ and a polynomial decay otherwise. This result improves that of Santos [69] in which some extra conditions on g'' and h'' were required. Mustafa [70] discussed (1.30) and gave sufficient conditions to guarantee the well-posedness of the system. In addition, under the following assumptions on the relaxation functions

$$\begin{aligned} g'(t) &\le -\xi_1(t) g(t), \quad t \ge 0 \\ h'(t) &\le -\xi_2(t) h(t), \quad t \ge 0, \end{aligned} \tag{1.31}$$

where $\xi_1, \xi_2 : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are non-increasing functions, he proved the existence and uniqueness result and established a generalized stability result from which exponential and polynomial decay rates are only special cases. Said-Houari *et al.* [71] considered a system of viscoelastic wave equations with nonlinear damping terms acting on both equations. Their work was mainly concerned with the following problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(s)ds + |u_t|^{m-1}u_t = f_1(u, v), & \text{in } \ \Omega \times (0, \infty), \\ v_{tt} - \Delta v + \displaystyle\int_0^t h(t-s)\Delta v(s)ds + |v_t|^{r-1}v_t = f_2(u, v), & \text{in } \ \Omega \times (0, \infty), \\ u = v = 0, & \text{on } \ \partial\Omega \times [0, \infty), \\ u(\cdot, 0) = u_0, \quad u_t(\cdot, 0) = u_1, \quad v(\cdot, 0) = v_0, \quad v_t(\cdot, 0) = v_1, & \text{in } \ \Omega, \end{cases} \tag{1.32}$$

with

$$f_1(u,v) = a|u+v|^{2(\rho+1)}(u+v) + b|u|^\rho u|v|^{\rho+2}$$

$$f_2(u,v) = a|u+v|^{2(\rho+1)}(u+v) + b|u|^{\rho+2}|v|^\rho v,$$

Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Under some conditions on the initial data, ρ, m, r, g, h, they proved the existence and uniqueness of local and global solutions. By imposing (1.31) on g and h, they established a generalized decay rate of the solution of (1.32). Their result improves the ones in Messaoudi and Tatar [68] and Liu [72]. Very recently, Al-Gharabli and Kafini [73] established a general decay result for (1.30) with the relaxation functions $g_i's$ satisfying

$$g_i'(t) \leq -G_i(g_i(t)), \qquad \forall\, t \geq 0, \qquad i = 1,\, 2 \tag{1.33}$$

with $G_i : [0,\infty) \longrightarrow [0,\infty)$ with $G_i(0) = 0$ and each G_i is linear or strictly increasing and strictly convex C^2 function on $(0, r]$ for some $r > 0$. This later result allowed a larger class of relaxation functions and generalizes, in some cases, the results in [68], [70], [69].

1.2 Results Description and Contributions

1.2.1 Results Description

Our main objective in this dissertation is to prove some general decay rates for viscoelastic-type Timoshenko and Bresse systems and a viscoelastic system of coupled wave equations with a wider class of relaxation functions. Our results extend that of Messaoudi and Al-Khulaifi [22] to the case of Timoshenko and Bresse systems with finite memories, and also, the result of Mustafa [23] to the case of system of viscoelastic wave equations. Precisely, the work is organized as follows.

In Chapter 2, we study the memory-type Timoshenko system of the form

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, & \text{in } (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \displaystyle\int_0^t g(t-s)\psi_{xx}(\cdot,s)ds = 0, & \text{in } (0,L)\times(0,+\infty), \\ \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, & \text{for } t \geq 0, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), & \text{for } x \in (0,L), \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), & \text{for } x \in (0,L), \end{cases} \tag{P_1}$$

where $b,\ K,\ \rho_1,\ \rho_2$ are positive constants, $\varphi_0,\ \varphi_1,\ \psi_0,\ \psi_1$ are given data and g is a relaxation function. We prove some general decay rates for the energy associated to the solution of (P_1) in the cases of equal and non-equal speeds of wave propagation by imposing the following assumptions on the relaxation function:

(**A.1**) $g : [0,\infty) \longrightarrow [0,\infty)$ is a nonincreasing and differentiable function with

$$g(0) > 0 \qquad \text{and} \qquad 1 - \int_0^\infty g(s)ds = l > 0.$$

(**A.2**) There exists a nonincreasing differentiable function $\xi : [0,\infty) \longrightarrow (0,\infty)$ satisfying, for some $1 \leq p < 2$,

$$g'(t) \leq -\xi(t)g^p(t), \qquad \forall\, t \geq 0.$$

In Chapter 3, we consider a finite-memory Bresse system of the form

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - lk_3(w_x - l\varphi) = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + lw) + \displaystyle\int_0^t g(t-s)\psi_{xx}(\cdot,s)ds = 0, \\ \rho_1 w_{tt} - lk_3(w_x - l\varphi)_x + lk_1(\varphi_x + \psi + lw) = 0, \\ \varphi(0,t) = \psi_x(0,t) = w_x(0,t) = \varphi(L,t) = \psi_x(L,t) = w_x(L,t) = 0, \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x), \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x), \end{cases} \tag{P_2}$$

where $(x,t) \in (0,L) \times \mathbb{R}_+$, $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a relaxation functions and L, l, ρ_i, k_i are positive constants. By imposing the assumptions (**A.1**) and (**A.2**) above and taking into consideration the nature of Bresse system, we prove, under a smallness condition on l, generalized energy decay results for the solution of (P_2) in the cases of equal and different speeds of wave propagation.

Chapter 4 deals with a coupled system of viscoelastic wave equations of the form

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g_1(t-s)\Delta u(\cdot,s)ds + f_1(u,v) = 0, & \text{in } \ \Omega \times (0,\infty), \\ v_{tt} - \Delta v + \displaystyle\int_0^t g_2(t-s)\Delta v(\cdot,s)ds + f_2(u,v) = 0, & \text{in } \ \Omega \times (0,\infty), \\ u = v = 0, & \text{on } \ \partial\Omega \times [0,\infty), \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, \quad v(\cdot,0) = v_0, \quad v_t(\cdot,0) = v_1, & \text{in } \ \Omega, \end{cases} \tag{P_3}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, u_0, v_0, u_1, v_1 are given initial data and f_i, g_i $(i = 1, 2)$ are given functions. The unknowns u and v represent the displacements of waves. This system can be considered as a generalization of the well-known Klein-Gordon system that appears in the quantum field theory. For more details, see [74], [75], [76]. We assume that the relaxation functions satisfy the following hypotheses:

(**A.3**) $g_i : [0, +\infty) \longrightarrow [0, +\infty)$ (for $i = 1, 2$) are non-increasing differentiable functions such that

$$g_i(0) > 0, \qquad 1 - \int_0^{+\infty} g_i(s)ds =: l_i > 0.$$

(**A.4**) There exist non-increasing differentiable functions $\xi_i \colon [0, +\infty) \longrightarrow (0, +\infty)$ and $\boldsymbol{C}^1-$functions $H_i : [0, +\infty) \longrightarrow [0, +\infty)$ which are linear or strictly increasing and strictly convex $\boldsymbol{C}^2-$functions on $(0, r]$, $r < g_i(0)$, with $H_i(0) = H_i'(0) = 0$ such that

$$g_i'(t) \leq -\xi_i(t) H_i(g_i(t)), \qquad \forall\, t \geq 0.$$

(**A.5**) $f_i : \mathbb{R}^2 \longrightarrow \mathbb{R}$ (for $i = 1, 2$) are $\boldsymbol{C}^1-$functions with $f_i(0, 0) = 0$ and there exists a function F such that

$$f_1(x, y) = \frac{\partial F}{\partial x}(x, y), \qquad f_2(x, y) = \frac{\partial F}{\partial y}(x, y),$$

$$F \geq 0, \qquad x f_1(x, y) + y f_2(x, y) - F(x, y) \geq 0,$$

and

$$\left|\frac{\partial f_i}{\partial x}(x,y)\right| + \left|\frac{\partial f_i}{\partial y}(x,y)\right| \leq d\left(1 + |x|^{\beta_i - 1} + |y|^{\beta_i - 1}\right), \quad \forall (x,y) \in \mathbb{R}^2, \tag{1.34}$$

for some constants $d > 0$ and

$$\beta_i \geq 1, \qquad \text{if } \ n = 1,2; \qquad 1 \leq \beta_i \leq \frac{n}{n-2}, \qquad \text{if } \ n \geq 3.$$

We prove a new general decay result for the solution of Problem (P_3).

1.2.2 Contributions

Timoshenko System

The energy decay rate for viscoelastic-type Timoshenko system has been established for relaxation function satisfying different conditions.

- In [33], Ammar-Khodja *et al.* assumed that the ralaxation function g satisfied, for any $t \geq 0$,

$$\exists k_0, k_1, k_2 > 0 : -k_0 g(t) \leq g'(t) \leq -k_1 g(t), \ |g''(t)| \leq k_2 g(t)$$

or

$$0 < g(t) \leq b_0(1+t)^{-p}, \qquad -b_1 g^{\frac{p+1}{p}}(t) \leq g'(t) \leq -b_2 g^{\frac{p+1}{p}}(t),$$
$$-b_3 |g'(t)|^{\frac{p+2}{p+1}} \leq g''(t) \leq -b_4 |g'(t)|^{\frac{p+2}{p+1}},$$

where k_0, k_1, k_2, b_0, b_1, b_2, b_3, b_4 and b_5 are positive constants.

- Guesmia and Messaoudi [34] imposed the following condition on g

$$g'(t) \leq -k_0 g^p(t), \qquad \forall\, t \geq 0,$$

where $1 \leq p < \frac{3}{2}$ and $k_0 > 0$ is a constant.

- Messaoudi and Mustafa [35] considered relaxation functions satisfying

$$g'(t) \leq -\xi(t) g(t), \qquad \forall\, t \geq 0,$$

where ξ is a differentiable nonincreasing positive function on $[0, \infty)$.

We proved our results under the following more general condition on g:

$$g'(t) \leq -\xi(t) g^p(t), \qquad \forall\, t \geq 0,$$

where ξ is a differentiable nonincreasing positive function on $[0, \infty)$ and $1 \leq p < 2$. Our result, for the case of equal-speeds of wave propagation, generalizes those in [33], [34], [35], it also improves that of Messaoudi and Mustafa [35] and gives better decay in the case of polynomial decay rate. Moreover, we discuss the non-equal speeds of wave propagation case and improve the result of Guesmia and Messaoudi [77].

Bresse System

We prove the energy decay rate for viscoelastic Bresse system with finite memory under the following condition on the relaxation function g:

$$g'(t) \leq -\xi(t)g^p(t), \qquad \forall\, t \geq 0,$$

where ξ is a differentiable nonincreasing positive function on $[0, \infty)$ and $1 \leq p < 2$. To the best of our knowledge, our result is the first in the literature that deals with the general decay rates for Bresse system with finite memory.

System of Viscoelastic Wave Equations

The energy decay rate for systems of viscoelastic wave equations has been established with different conditions on the relaxation functions g_i ($i = 1,\, 2$).

- In [69], Santos assumed that the ralaxation functions g_i satisfied, for $i = 1, 2$ and any $t \geq 0$,

  $$-c_0 g_i(t) \leq g_i'(t) \leq -c_1 g_i(t), \quad 0 \leq g_i''(t) \leq c_2 g_i(t),$$

  where c_0, c_1, c_2, are some positive constants.

- Messaoudi and Tatar [68] imposed the following condition on g_i

  $$g_i'(t) \leq -\xi_i g_i^p(t), \qquad \forall\, t \geq 0,$$

where $1 \leq p < \frac{3}{2}$ and $\xi_i > 0$ is a constant, for $i = 1, 2$.

- Mustafa [70] consider relaxation functions of the form

$$g_i'(t) \leq -\xi_i(t) g_i(t), \qquad \forall\, t \geq 0,$$

where ξ_i, for $i = 1, 2$, is a differentiable nonincreasing positive function on $[0, \infty)$

.

- Al-Gharabli and Kafini [73] established their results with the following relaxation functions:

$$g_i(t) \leq -H_i(g_i(t)) \qquad \forall\, t \geq 0.$$

We proved our results under the following condition on g_i:

$$g_i'(t) \leq -\xi_i(t) H_i(g_i(t)), \qquad \forall\, t \geq 0,$$

where ξ_i, for $i = 1, 2$, is a differentiable nonincreasing positive function on $[0, \infty)$ and H_i is a linear or strictly increasing and strictly convex function on $[0, \infty)$ satisfying some additional conditions to be specified later. Our result generalize those in [68], [69], [70], [73], it also improves that of Mustafa [70] and allow wider classes of relaxation functions.

1.3 Methodology

We established the results in this work by using convexity and multiplier methods. For problems (P_1) and (P_2), we start by carefully constructing a Lypunov functional $\mathcal{L}$ that is equivalent to the energy associated to the solution of each problem, respectively, in the case of equal speeds of wave propagation. By equivalence of the Lyapunov functional $\mathcal{L}$ and the energy functional E, we mean the existence of two positive real numbers α_1, α_2 such that

$$\alpha_1 \mathcal{L}(t) \leq E(t) \leq \alpha_2 \mathcal{L}(t), \qquad \forall\, t \geq 0.$$

The relaxation function g in problem (P_1) and (P_2) satisfies the following hypotheses:

(A.1) $g : [0, \infty) \longrightarrow (0, \infty)$ is a non-increasing and differentiable function with

$$g(0) > 0 \qquad \text{and} \qquad 1 - \int_0^\infty g(s)ds = l > 0.$$

(A.2) There exists a non-increasing differentiable function $\xi : [0, \infty) \longrightarrow (0, \infty)$ and a constant p, with $1 \leq p < 2$, such that

$$g'(t) \leq -\xi(t) g^p(t), \qquad \forall\, t \geq 0.$$

Then, we show that our Lyapunov functional satisfies, for any $t_0 > 0$, the following estimate

$$\mathcal{L}'(t) \leq -kE(t) + c(g \circ \psi_x)(t), \qquad \forall\, t \geq t_0, \tag{1.35}$$

where k is a fixed positive constant, c is a generic constant and $g \circ v$ is defined by

$$(g \circ v)(t) := \int_0^t g(t-s)\|v(t) - v(s)\|_2^2 ds.$$

Next, we use Jensen's inequality to show that

$$\mathcal{L}'(t) \leq -mE(t) + c\left(-\frac{E'(t)}{\xi(t)}\right)^{\frac{1}{p}}, \qquad \forall\, t \geq t_0.$$

We exploit the non-increasing properties of E and ξ to arrive at

$$\xi(t)\big(E^{p-1}\mathcal{L}\big)'(t) \leq -m_0\xi(t)E(t) - cE'(t), \qquad \forall\, t \geq t_0,$$

from which, we obtain the energy decay rate

$$E(t) \leq C\left(1 + \int_{t_0}^t \xi ds\right)^{-\frac{1}{p-1}}, \qquad \forall\, t > t_0.$$

In the case of non-equal speeds of wave propagation, we construct a functional $\mathcal{F}_2$ satisfying

$$\mathcal{F}_2 \geq cE + cE_*, \qquad \forall\, t \geq 0$$

and

$$\mathcal{F}_2'(t) \leq -m_2 t\xi(t)\left(\frac{E(t)}{t}\right)^p + c_2 g(t), \qquad \forall\, t \geq 0$$

from this we arrive at the decay rate of the form

$$E(t) \leq C t^{1-1/p}\left(\int_{t_0}^{t}\xi(s)ds\right), \qquad \forall\, t > t_0.$$

For Problem (P_3), the relaxation functions are assumed to satisfy the following conditions:

(**A.3**) $g_i : [0, +\infty) \longrightarrow (0, +\infty)$ (for $i = 1, 2$) are non-increasing differentiable functions such that

$$g_i(0) > 0, \qquad 1 - \int_0^{+\infty} g_i(s)ds =: l_i > 0.$$

(**A.4**) There exist non-increasing differentiable functions $\xi_i : [0, +\infty) \longrightarrow (0, +\infty)$ and $\boldsymbol{C}^1$ functions $H_i : [0, +\infty) \longrightarrow [0, +\infty)$ which are linear or strictly increasing and strictly convex $\boldsymbol{C}^2$ functions on $(0, r]$, $r \leq g_i(0)$, with $H_i(0) = H_i'(0) = 0$ such that

$$g_i'(t) \leq -\xi_i(t) H_i(g_i(t)), \qquad \forall\, t \geq 0 \quad \text{and for } i = 1, 2.$$

(**A.5**) $f_i : \mathbb{R}^2 \longrightarrow \mathbb{R}$ (for $i = 1, 2$) are $\boldsymbol{C}^1$ functions with $f_i(0, 0) = 0$ and there exists a function F such that

$$f_1(x, y) = \frac{\partial F}{\partial x}(x, y), \qquad f_2(x, y) = \frac{\partial F}{\partial y}(x, y),$$

$$F \geq 0, \qquad x f_1(x,y) + y f_2(x,y) - F(x,y) \geq 0,$$

and

$$\left|\frac{\partial f_i}{\partial x}(x,y)\right| + \left|\frac{\partial f_i}{\partial y}(x,y)\right| \leq d\left(1 + |x|^{\beta_i - 1} + |y|^{\beta_i - 1}\right), \quad \forall (x,y) \in \mathbb{R}^2, \qquad (1.36)$$

for some constants $d > 0$ and

$$\beta_i \geq 1, \qquad \text{if } \; n = 1, 2; \qquad 1 \leq \beta_i \leq \frac{n}{n-2}, \qquad \text{if } \; n \geq 3.$$

We start by constructing a Lyapunov functional $\mathcal{F}$ that is equivalent to the energy E associated to the solution of the system and satisfies, for any fixed $t_0 > 0$, the estimate

$$\begin{aligned} \mathcal{F}'(t) \;\; &\leq \;\; -mE(t) + c\int_{t_0}^{t} g_1(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds \\ &\quad\; +c\int_{t_0}^{t} g_2(s)\|\nabla v(t) - \nabla v(t-s)\|_2^2 ds, \qquad \forall\, t \geq t_0. \qquad (1.37) \end{aligned}$$

Then, we define other functionals θ_i

$$\theta_1(t) := -\int_{t_0}^{t} g_1'(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds,$$

$$\theta_2(t) := -\int_{t_0}^{t} g_2'(s)\|\nabla v(t) - \nabla v(t-s)\|_2^2 ds.$$

By exploiting the dissipativeness of E, convexity of H_i and Jensen's inequality we establish

$$\mathcal{F}'(t) \leq -mE(t) + c\bar{H}_1^{-1}\left(\frac{\gamma\theta_1(t)}{\xi_1(t)}\right) + c\bar{H}_2^{-1}\left(\frac{\gamma\theta_2(t)}{\xi_2(t)}\right), \qquad \forall\, t \geq t_0. \tag{1.38}$$

Next, we set $G = \min\{\bar{H}_1', \bar{H}_2'\}$ and, for a fixed $0 < \varepsilon < r$, we define a functional $\mathcal{F}_1$ by

$$\mathcal{F}_1(t) := G\left(\varepsilon\frac{E(t)}{E(0)}\right)\mathcal{F}(t) + E(t), \qquad \forall\, t \geq 0.$$

We use estimate (1.37) and the generalized Young inequality to obtain

$$E(t) \leq k_2 G_*^{-1}\left(k_1 \int_{t_0}^{t} \xi(s)ds\right) \qquad \forall\, t > t_0,$$

where $G_*(t) := \displaystyle\int_t^r \frac{1}{sG(s)}ds$.

1.4 Some Important Notations and Inequalities

In this Section, we present some notations and inequalities that are frequently used throughout this work.

- The sysmbol $\mathbb{R}_+$ denotes the set of non-negative real numbers, that is, $\mathbb{R}_+ = [0, +\infty)$.

- Let $\Omega \subset \mathbb{R}^N$ and m be a non-negative integer, then we have the following:

  $\boldsymbol{C}^m(\Omega)$ represents the space of $m-$times continuously differentiable functions defined on Ω,

  $\boldsymbol{C}_0^m(\Omega)$ represents the space of $m-$times continuously differentiable functions with compact support in Ω,

$$\boldsymbol{C}^{\infty}(\Omega)=\bigcap_{m\geq 0}\boldsymbol{C}^{m}(\Omega) \qquad \text{and} \qquad \boldsymbol{C}_0^{\infty}(\Omega)=\bigcap_{m\geq 0}\boldsymbol{C}_0^{m}(\Omega).$$

- Let $\Omega\subset\mathbb{R}^N$ be a domain, we denote by $L^p(\Omega)$, $1\leq p<+\infty$, the Lebesgue space

$$L^p(\Omega):=\left\{f:\Omega\longrightarrow\mathbb{R}\Big|f \;\text{ is measurable and }\; \int_{\Omega}|f|^p<+\infty\right\},$$

equipped with the norm

$$\|f\|_p=\|f\|_{L^p(\Omega)}=\left(\int_{\Omega}|f(x)|^p dx\right)^{1/p}.$$

The space $L^{\infty}(\Omega)$ denotes

$$L^{\infty}(\Omega):=\left\{f:\Omega\longrightarrow\mathbb{R}\Big|f \;\text{ is measurable and }\; \exists M\geq 0 \;\text{ s.t }\; |f|\leq M \;\; a.e. \;\text{ on }\; \Omega\right\},$$

equipped with the norm

$$\|f\|_{\infty}=\|f\|_{L^{\infty}(\Omega)}=\inf\{M\geq 0:|f(x)|\leq M \quad a.e. \quad \text{on} \quad \Omega\}.$$

- $L^p_{loc}(\Omega)=\big\{f:\Omega\longleftarrow\mathbb{R}\big|\; f \;\text{ is measurable and }\; f\in L^p(K)\;\forall\,K\hookrightarrow\hookrightarrow\Omega\big\}$

- For $m\in\mathbb{N}$, $1\leq p\leq\infty$, we define the Sobolev space

$$W^{m,p}(\Omega)=\left\{u\in L^p(\Omega)\Big|\; D^{\alpha}u\in L^p(\Omega) \;\;\forall\,\alpha\in\mathbb{N}^N \;\text{ with }\; |\alpha|\leq m\right\}$$

equipped with the norm

$$\|u\|_{m,p} = \|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \le m} \|D^\alpha u\|_p^p \right)^{1/p}.$$

$D^\alpha u$ is the $\alpha-$th "weak" partial derivative of u which is defined as a locally integrable function g satisfying

$$\int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega g\varphi \qquad \forall\, \varphi \in \boldsymbol{C}_0^\infty(\Omega),$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$ with $\alpha_i \ge 0$ an integer,

$$|\alpha| = \alpha_1 + \cdots + \alpha_N \qquad \text{and} \qquad D^\alpha u = \frac{\partial^{\alpha_1 + \cdots + \alpha_N} u}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}.$$

The space $W^{m,2}(\Omega)$ is denoted by $H^m(\Omega)$ which is a Hilbert space.

- The space $W_0^{m,p}(\Omega)$ denotes the closure of $\boldsymbol{C}_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$ with respect to $W^{m,p}(\Omega)$ norm. The space $W_0^{m,2}(\Omega)$ is denoted by $H_0^m(\Omega)$.
- $\nabla u = \left(\dfrac{\partial u}{\partial x_1}, \cdots, \dfrac{\partial u}{\partial x_N} \right)$, for any $u \in W^{1,p}(\Omega)$
- $\Delta u = \displaystyle\sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}$, for any $u \in W^{2,p}(\Omega)$
- $u_t = \dfrac{\partial u}{\partial t}, \quad u_{tt} = \dfrac{\partial^2 u}{\partial t^2}$
- Let $(X, \|\cdot\|_X)$ be a Banach space, $1 \le p < \infty$, $-\infty \le a < b \le \infty$. Then,

  1. $L^p\big((a,b); X\big)$ denotes the space of L^p functions from (a,b) into X equipped

with the norm

$$\|f\|_{L^p((a,b);X)} = \left(\int_a^b \|f(t)\|_X^p dt \right)^{1/p}.$$

2. For $p = \infty$, $L^\infty\big((a,b);X\big)$ denotes the space of measurable functions from (a,b) into X that are essentially bounded, it is equipped with the norm

$$\|f\|_{L^p((a,b);X)} = \operatorname*{ess\,sup}_{t\in(a,b)} \|f(t)\|_X.$$

3. Similarly, for $-\infty < a < b < \infty$, we defined $\boldsymbol{C}^m\big([a,b];X\big)$ equipped with the norm

$$\|f\|_{\boldsymbol{C}([a,b];X)} = \sum_{i=0}^m \max_{t\in[a,b]} \|f^{(i)}(t)\|_X.$$

Throughout this work, we use the symbols c and C to denote generic positive constants that vary from one occurrence to the next. We also use the following inequalities repeatedly.

1. **Young's inequality.** Let $1 < p, q < \infty$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then for any $\varepsilon > 0$, we have

$$ab \le \varepsilon a^p + C_\varepsilon b^q \qquad \forall\, a,b > 0,$$

where $C_\varepsilon = \dfrac{1}{q(\varepsilon p)^{p/q}}$. For $p = q = 2$, we have

$$ab \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2.$$

2. **Hölder's inequality.** Let $1 \le p, q \le \infty$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. If $u \in L^p(\Omega)$ and

$v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$

$$\int_\Omega |uv| \le \|u\|_p \|v\|_q.$$

By taking $p = q = 2$, we obtain **Cauchy-Schwarz inequality**.

3. **Poincaré's inequality.** Suppose $1 \le p < \infty$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain of class $\boldsymbol{C}^1$. Then there is a positive constant C depending only on Ω and N such that

$$\|u\|_p \le C\|\nabla u\|_p \qquad \forall\, u \in W_0^{1,p}(\Omega) \cup W_*^{1,p}(\Omega),$$

where $W_*^{1,p}(\Omega) := \left\{ u \in W^{1,p}(\Omega) : \int_\Omega u = 0 \right\}$.

4. **Jensen's inequality.**

(i) Let $G : [a,b] \longrightarrow \mathbb{R}$ be a convex function. Assume that the functions $f : \Omega \longrightarrow [a,b]$ and $h : \Omega \longrightarrow \mathbb{R}$ are integrable such that $h(x) \ge 0$, for any $x \in \Omega$ and $\int_\Omega h(x)dx = k > 0$. Then,

$$G\left(\frac{1}{k}\int_\Omega f(x)h(x)dx\right) \le \frac{1}{k}\int_\Omega G(f(x))h(x)dx.$$

(ii) Let $G : [a,b] \longrightarrow \mathbb{R}$ be a concave function. Assume that the functions $f : \Omega \longrightarrow [a,b]$ and $h : \Omega \longrightarrow \mathbb{R}$ are integrable such that $h(x) \ge 0$, for any $x \in \Omega$ and $\int_\Omega h(x)dx = k > 0$. Then,

$$\frac{1}{k}\int_\Omega G(f(x))h(x)dx \le G\left(\frac{1}{k}\int_\Omega f(x)h(x)dx\right).$$

In particular, for $G(y) = y^{\frac{1}{p}},\ y \geq 0,\ p > 1$, we have

$$\frac{1}{k}\int_\Omega f^{1/p}(x)h(x)dx \leq \left(\frac{1}{k}\int_\Omega f(x)h(x)dx\right)^{1/p}.$$

5. **Green's formula.** Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a regular boundary $\partial\Omega$. Then

$$\int_\Omega u\Delta v dx + \int_\Omega \nabla u \cdot \nabla v dx = \int_{\partial\Omega} u\frac{\partial u}{\partial n}d\sigma, \qquad \forall\, u \in H^1(\Omega),\ v \in H^2(\Omega).$$

CHAPTER 2

GENERAL AND OPTIMAL DECAY IN A MEMORY-TYPE TIMOSHENKO SYSTEM

Our main purpose in this chapter is to study the following memory-type Timoshenko system

$$
\begin{cases}
\rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, & \text{in } (0,L)\times(0,+\infty),\\
\rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x+\psi) + \displaystyle\int_0^t g(t-s)\psi_{xx}(s)ds = 0, & \text{in } (0,L)\times(0,+\infty),\\
\varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, & \text{for } t\geq 0,\\
\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), & \text{for } x\in(0,L),\\
\psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), & \text{for } x\in(0,L),
\end{cases}
\tag{P_1}
$$

where $b,\ K,\ \rho_1,\ \rho_2$ are positive constants, $\varphi_0,\ \varphi_1,\ \psi_0,\ \psi_1$ are given data and g is a relaxation function. This chapter is organized as follows: in Section 2.1, we state some preliminary results. In Section 2.2, we state and prove some technical lemmas. The statement and proof of our main results are given in Sections 2.3 and 2.4.

2.1 Preliminaries

In this section, we present some useful lemmas and state the existence theorem.

Lemma 2.1 (Muñoz Rivera and Racke [78]) *For* $g, w \in \boldsymbol{C}^1\big(\mathbb{R}_+;\mathbb{R}\big)$ *we have*

$$2w'(t)\int_0^t g(t-s)w(s)ds = \frac{d}{dt}\left[\left(\int_0^t g(s)ds\right)|w(t)|^2 - (g\circ w)(t)\right] \\ - g(t)|w(t)|^2 + (g'\circ w)(t).$$

Definition 2.1 *A pair* $(\varphi,\ \psi)$ *of functions defined on* $[0,T]\times(0,L)$ *is said be to a weak solution of* (P_1) *if*

$$\varphi,\ \psi \in \boldsymbol{C}\big([0,T];H_0^1(0,L)\big)\cap\boldsymbol{C}^1\big([0,T];L^2(0,L)\big)\cap\boldsymbol{C}^2\big([0,T];H^{-1}(0,L)\big),$$

$$\big(\varphi(\cdot,0),\ \psi(\cdot,0)\big) = \big(\varphi_0,\ \psi_0\big)\in H_0^1(0,L)\times H_0^1(0,L),$$

$$\big(\varphi_t(\cdot,0),\ \psi_t(\cdot,0)\big) = \big(\varphi_1,\ \psi_1\big)\in L^2(0,L)\times L^2(0,L)$$

and (φ,ψ) *satisfies*

$$\rho_1\int_0^L \varphi_t u dx - \rho_1\int_0^L \varphi_1 u dx + K\int_0^t\int_0^L(\varphi_x+\psi)u'dx = 0,$$

$$\rho_2 \int_0^L \psi_t v dx - \rho_2 \int_0^L \psi_1 v dx + b \int_0^t \int_0^L \psi_x v' dx + K \int_0^t \int_0^L (\varphi_x + \psi) v dx$$
$$- \int_0^t \int_0^L \int_0^\tau g(\tau - s) \psi_x(s) v' ds dx d\tau = 0,$$

for any $(u, v) \in H_0^1(0, L) \times \in H_0^1(0, L)$ *and any* $t \in [0, T]$*. Here,* $'$ *denotes a derivative with respect to* x*.*

In addition, if (φ, ψ) *satisfies*

$$\varphi, \psi \in \boldsymbol{C}([0, T]; H^2(0, L) \cap H_0^1(0, L)) \cap \boldsymbol{C}^1([0, T]; H_0^1(0, L)) \cap \boldsymbol{C}^2([0, T]; L^2(0, L)),$$

with

$$(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in \left(H^2(0, L) \cap H_0^1(0, L)\right) \times H_0^1(0, L),$$

then (φ, ψ) *is said to be a strong solution of* (P_1)*.*

Theorem 2.1 *Let* $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in H_0^1(0, L) \times L^2(0, L)$ *be given. Assume that* g *satisfies hypothesis (**A.1**). Then, problem* (P_1) *has a unique global (weak) solution*

$$\varphi, \psi \in \boldsymbol{C}(\mathbb{R}_+; H_0^1(0, L)) \cap \boldsymbol{C}^1(\mathbb{R}_+; L^2(0, L)).$$

Moreover, if $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in (H^2(0, L) \cap H_0^1(0, L)) \times H_0^1(0, L)$*, then the problem has a unique strong solution*

$$\varphi, \psi \in \boldsymbol{C}(\mathbb{R}_+; H^2(0, L) \cap H_0^1(0, L)) \cap \boldsymbol{C}^1(\mathbb{R}_+; H_0^1(0, L)) \cap \boldsymbol{C}^2(\mathbb{R}_+; L^2(0, L)).$$

Proof. Let $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in (H^2(0, L) \cap H_0^1(0, L)) \times H_0^1(0, L)$ and let $\{w_j\}_{j \geq 1}$

be an orthogonal basis for $H_0^1(0,L)$ and $H^2(0,L)\cap H_0^1(0,L)$ which is orthonormal in $L^2(0,L)$ with

$$w_j'' = -\lambda_j w_j \qquad \text{on} \quad (0,L), \quad j \geq 1.$$

By elliptic regularity, $w_j \in C^\infty\big([0,L]\big)\cap H_0^1(0,L)$. Define a sequence of finite dimensional spaces V_m by

$$V_m := \operatorname{span}\{w_1, w_2, \dots, w_m\}, \quad m \geq 1.$$

We seek for approximate solutions of the form

$$\varphi^m(x,t) := \sum_{j=1}^{m} a_{j,m}(t)w_j(x), \qquad \psi^m(x,t) := \sum_{j=1}^{m} b_{j,m}(t)w_j(x)$$

to the following approximate problems in V_m

$$\begin{cases} \rho_1 \displaystyle\int_0^L \varphi_{tt}^m u dx + K\int_0^L (\varphi_x^m + \psi^m)u' dx = 0, \\ \\ \rho_2 \displaystyle\int_0^L \psi_{tt}^m v dx + b\int_0^L \psi_x^m v' dx + K\int_0^L (\varphi_x^m + \psi^m) v dx \\ \qquad\qquad\qquad \displaystyle - \int_0^L \int_0^t g(t-s)\psi_x^m(\cdot,s) v' ds dx = 0, \\ \\ \varphi^m(\cdot,0) = \varphi_0^m,\ \varphi_t^m(\cdot,0) = \varphi_1^m,\ \psi^m(\cdot,0) = \psi_0^m,\ \psi_t^m(\cdot,0) = \psi_1^m, \end{cases} \tag{2.1}$$

where $u,\, v \in V_m$ and

$$
\begin{cases}
\varphi_0^m := \sum_{j=1}^{m} (\varphi_0, w_j)_{L^2(0,L)} w_j \;\longrightarrow\; \varphi_0 & \text{in } \; H^2(0,L) \cap H_0^1(0,L), \\
\psi_0^m := \sum_{j=1}^{m} (\psi_0, w_j)_{L^2(0,L)} w_j \;\longrightarrow\; \psi_0 & \text{in } \; H^2(0,L) \cap H_0^1(0,L), \\
\varphi_1^m := \sum_{j=1}^{m} (\varphi_1, w_j)_{L^2(0,L)} w_j \;\longrightarrow\; \varphi_1 & \text{in } \; H_0^1(0,L), \\
\psi_1^m := \sum_{j=1}^{m} (\psi_1, w_j)_{L^2(0,L)} w_j \;\longrightarrow\; \psi_1 & \text{in } \; H_0^1(0,L).
\end{cases}
\tag{2.2}
$$

This leads to a system of linear ordinary differential equations (ODEs) with the unknown functions $a_{j,m}$ and $b_{j,m}$. Standard ODE theory guarantees the existence of unique $\boldsymbol{C}^2-$solution $(\varphi^m,\, \psi^m)$ on the maximal interval $[0, t_m)$ for each $m \geq 1$.

The next a priori estimate shows that $t_m = \infty$ for any $m \geq 1$.

First a priori estimate: Let $u = \varphi_t^m$ in $(2.1)_1$, $v = \psi_t^m$ in $(2.1)_2$, exploit Lemma 2.1 and then add the resultants to get, for any $t \geq 0$,

$$
\begin{aligned}
\frac{d}{dt}\frac{1}{2}\Bigg[& \rho_1 \|\varphi_t^m(t)\|_2^2 + K\|\varphi_x^m(t) + \psi^m(t)\|_2^2 + \rho_2\|\psi_t^m(t)\|_2^2 \\
& + \left(b - \int_0^t g(s)ds \right) \|\psi_x^m(t)\|_2^2 + (g \circ \psi_x^m)(t) \Bigg] \\
& = -\frac{1}{2} g(t)\|\psi_x^m(t)\|_2^2 + \frac{1}{2}(g' \circ \psi_x^m)(t) \leq 0.
\end{aligned}
$$

Integration over $(0,t)$ yields, for any $m \geq 1$ and $t \geq 0$,

$$\begin{aligned}
&\frac{1}{2}\Bigg[\rho_1\|\varphi_t^m(t)\|_2^2 + K\|\varphi_x^m(t) + \psi^m(t)\|_2^2 + \rho_2\|\psi_t^m(t)\|_2^2 \\
&\quad + \left(b - \int_0^t g(s)ds\right)\|\psi_x^m(t)\|_2^2 + (g \circ \psi_x^m)(t)\Bigg] \\
&\quad \leq \frac{1}{2}\left[\rho_1\|\varphi_1^m\|_2^2 + K\|(\varphi_0^m)' + \psi_0^m\|_2^2 + \rho_2\|\psi_1^m\|_2^2 + b\|(\psi_0^m)'\|_2^2\right] \\
&\quad \leq \frac{1}{2}\left[\rho_1\|\varphi_1\|_2^2 + K\|\varphi_0' + \psi_0\|_2^2 + \rho_2\|\psi_1\|_2^2 + b\|\psi_0'\|_2^2\right] \leq C,
\end{aligned} \tag{2.3}$$

where $C > 0$ is a constant independent of m and t.

Second a priori estimate: Let $u = -\varphi_{xxt}^m$ in $(2.1)_1$, $v = -\psi_{xxt}^m$ in $(2.1)_2$, use Lemma 2.1 and then add the resultants to get, for any $m \geq 1$ and $t \geq 0$,

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\Bigg[\rho_1\|\varphi_{xt}^m(t)\|_2^2 + K\|(\varphi_x^m(t) + \psi^m)_x(t)\|_2^2 + \rho_2\|\psi_{xt}^m(t)\|_2^2 \\
&\quad + \left(b - \int_0^t g(s)ds\right)\|\psi_{xx}^m(t)\|_2^2 + (g \circ \psi_{xx}^m)(t)\Bigg] \\
&\quad = -\frac{1}{2}g(t)\|\psi_{xx}^m(t)\|_2^2 + \frac{1}{2}(g' \circ \psi_{xx}^m)(t) \leq 0.
\end{aligned}$$

Integration over $(0,t)$ yields, for any $m \geq 1$ and $t \geq 0$,

$$\begin{aligned}
&\frac{1}{2}\Bigg[\rho_1\|\varphi_{xt}^m(t)\|_2^2 + K\|(\varphi_x^m(t) + \psi^m)_x(t)\|_2^2 + \rho_2\|\psi_{xt}^m(t)\|_2^2 \\
&\quad + \left(b - \int_0^t g(s)ds\right)\|\psi_{xx}^m(t)\|_2^2 + (g \circ \psi_{xx}^m)(t)\Bigg] \\
&\quad \leq \frac{1}{2}\left[\rho_1\|(\varphi_1^m)'\|_2^2 + K\|(\varphi_0^m)'' + (\psi_0^m)'\|_2^2 + \rho_2\|(\psi_1^m)'\|_2^2 + b\|(\psi_0^m)''\|_2^2\right] \leq C.
\end{aligned} \tag{2.4}$$

Third a priori estimate: Let $u = \varphi_{tt}^m$ in $(2.1)_1$, $v = \psi_{tt}^m$ in $(2.1)_2$, apply Lemma

2.1, Young's inequality, (2.3) and (2.4) to get, for any $m \geq 1$ and $t \geq 0$,

$$\|\varphi_{tt}^m\|_2^2 + \|\psi_{tt}^m\|_2^2 \leq C. \tag{2.5}$$

Thus, we deduce from (2.3)–(2.5) that

$$\begin{aligned}
&\big(\varphi^m\big) \quad \text{and} \quad \big(\psi^m\big) \quad \text{are bounded in} \quad L^\infty\big((0,\infty); H^2(0,L) \cap H_0^1(0,L)\big),\\
&\big(\varphi_t^m\big) \quad \text{and} \quad \big(\psi_t^m\big) \quad \text{are bounded in} \quad L^\infty\big((0,\infty); H_0^1(0,L)\big),\\
&\big(\varphi_{tt}^m\big) \quad \text{and} \quad \big(\psi_{tt}^m\big) \quad \text{are bounded in} \quad L^\infty\big((0,\infty); L^2(0,L)\big),
\end{aligned} \tag{2.6}$$

Therefore, we can extract a subsequence $\big(\varphi^k, \psi^k\big)$ of $\big(\varphi^m, \psi^m\big)$ such that

$$\begin{aligned}
&\varphi^k \longrightarrow \varphi \quad \text{and} \quad \psi^k \longrightarrow \psi \quad \text{weakly star in} \quad L^\infty\big((0,\infty); H^2(0,L) \cap H_0^1(0,L)\big),\\
&\varphi_t^k \longrightarrow \varphi_t \quad \text{and} \quad \psi_t^k \longrightarrow \psi_t \quad \text{weakly star in} \quad L^\infty\big((0,\infty); H_0^1(0,L)\big),\\
&\varphi_{tt}^k \longrightarrow \varphi_{tt} \quad \text{and} \quad \psi_{tt}^k \longrightarrow \psi_{tt} \quad \text{weakly star in} \quad L^\infty\big((0,\infty); L^2(0,L)\big).
\end{aligned} \tag{2.7}$$

In particular, for any fixed $T > 0$, we have:

$$\begin{aligned}
&\varphi^k \longrightarrow \varphi \quad \text{and} \quad \psi^k \longrightarrow \psi \quad \text{weakly in} \quad L^2\big((0,T); H^2(0,L) \cap H_0^1(0,L)\big),\\
&\varphi_t^k \longrightarrow \varphi_t \quad \text{and} \quad \psi_t^k \longrightarrow \psi_t \quad \text{weakly in} \quad L^2\big((0,T); H_0^1(0,L)\big),\\
&\varphi_{tt}^k \longrightarrow \varphi_{tt} \quad \text{and} \quad \psi_{tt}^k \longrightarrow \psi_{tt} \quad \text{weakly in} \quad L^2\big((0,T); L^2(0,L)\big).
\end{aligned} \tag{2.8}$$

Replacing m by k in $(2.1)_1$ and $(2.1)_2$ and then integrating over $(0,t)$ for any $t \in (0,T)$

we obtain, for $1 \leq j \leq k$,

$$\rho_1 \int_0^L \varphi_t^k w_j dx - \int_0^L \varphi_1^k w_j dx + K \int_0^t \int_0^L (\varphi_x^k + \psi^k) w_j' dx = 0,$$
$$\rho_2 \int_0^L \psi_t^k w_j dx - \int_0^L \psi_1^k w_j dx + b \int_0^t \int_0^L \psi_x^k w_j' dx + K \int_0^t \int_0^L (\varphi_x^k + \psi^k) w_j dx$$
$$- \int_0^t \int_0^L \int_0^\tau g(\tau - s) \psi_x^k(\cdot, s) w_j' ds dx d\tau = 0.$$

As k goes to infinity we have, for any $j \geq 1$,

$$\rho_1 \int_0^L \varphi_t w_j dx - \int_0^L \varphi_1 w_j dx + K \int_0^t \int_0^L (\varphi_x + \psi) w_j' dx = 0,$$
$$\rho_2 \int_0^L \psi_t w_j dx - \int_0^L \psi_1 w_j dx + b \int_0^t \int_0^L \psi_x w_j' dx + K \int_0^t \int_0^L (\varphi_x + \psi) v dx$$
$$- \int_0^t \int_0^L \int_0^\tau g(\tau - s) \psi_x(\cdot, s) w_j' ds dx d\tau = 0.$$

Thus, for any $(u, v) \in H_0^1(0, L) \times H_0^1(0, L)$, we get

$$\rho_1 \int_0^L \varphi_t u dx - \int_0^L \varphi_1 u dx + K \int_0^t \int_0^L (\varphi_x + \psi) u' dx d\tau = 0,$$
$$\rho_2 \int_0^L \psi_t v dx - \int_0^L \psi_1 v dx + b \int_0^t \int_0^L \psi_x v' dx d\tau + K \int_0^t \int_0^L (\varphi_x + \psi) v dx d\tau$$
$$- \int_0^t \int_0^L \int_0^\tau g(\tau - s) \psi_x(\cdot, s) v' ds dx d\tau = 0.$$

Differentiating both sides of the above with respect to t and integrating by parts with respect to x, we obtain

$$\rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0, \qquad \text{in} \quad L^2\big((0, T); L^2(0, L)\big),$$
$$\rho_2 \psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) - \int_0^t g(t - s) \psi_{xx}(\cdot, s) ds = 0, \qquad \text{in} \quad L^2\big((0, T); L^2(0, L)\big).$$

Next, we show that φ and ψ satisfy the initial conditions. Using Aubin-Lions Lemma, it follows from (2.7) that

$$\varphi^k \longrightarrow \varphi \quad \text{and} \quad \psi^k \longrightarrow \psi \quad \text{in} \quad \boldsymbol{C}\big([0,T]; H_0^1(0,L)\big),$$
$$\varphi_t^k \longrightarrow \varphi_t \quad \text{and} \quad \psi_t^k \longrightarrow \psi_t \quad \text{in} \quad \boldsymbol{C}\big([0,T]; L^2(0,L)\big).$$

Therefore, $\varphi^k(\cdot,0)$, $\psi^k(\cdot,0)$, $\varphi_t^k(\cdot,0)$, $\psi_t^k(\cdot,0)$ make sense and

$$\varphi^k(\cdot,0) \longrightarrow \varphi(\cdot,0) \quad \text{and} \quad \psi^k(\cdot,0) \longrightarrow \psi(\cdot,0) \quad \text{in} \quad H_0^1(0,L),$$
$$\varphi_t^k(\cdot,0) \longrightarrow \varphi_t(\cdot,0) \quad \text{and} \quad \psi_t^k(\cdot,0) \longrightarrow \psi_t(\cdot,0) \quad \text{in} \quad L^2(0,L).$$

A combination of these and (2.2) yields

$$\varphi(\cdot,0) = \varphi_0, \quad \psi(\cdot,0) = \psi_0, \quad \varphi_t(\cdot,0) = \varphi_1 \quad \text{and} \quad \psi_t(\cdot,0) = \psi_1.$$

Hence, a pair (φ,ψ) is a strong solution of (P_1).

For the uniqueness, assume that (φ,ψ) and $(\tilde{\varphi},\tilde{\psi})$ are pairs of weak solutions to Problem (P_1), then the pair $(\Phi,\Psi) = (\varphi-\tilde{\varphi},\psi-\tilde{\psi})$ satisfies, for any $(u,v) \in H_0^1(0,L)\times H_0^1(0,L)$,

$$\begin{aligned}
&\rho_1\int_0^L \Phi_{tt} u dx + K\int_0^L (\Phi_x+\Psi)u' dx = 0,\\
&\rho_2\int_0^L \Psi_{tt} v dx + b\int_0^L \Psi_x v' dx + K\int_0^L (\Phi_x+\Psi) v dx\\
&\qquad\qquad - \int_0^L\int_0^t g(t-s)\Psi_x(\cdot,s) v' ds dx d = 0,
\end{aligned}$$

$$\Phi(\cdot,0) = \Phi_t(\cdot,0) = \Psi(\cdot,0) = \Psi_t(\cdot,0) = 0.$$

Replacing (u, v) by (Φ_t, Ψ_t), exploiting Lemma 2.1 and adding the resultants, we have

$$\frac{d}{dt}\frac{1}{2}\Bigg[\rho_1\|\Phi_t\|_2^2 + K\|\Phi_x + \Psi\|_2^2 + \rho_2\|\Psi_t\|_2^2 + b\|\Psi_x\|_2^2 + (g \circ \Psi_x)(t)\Bigg] = -\frac{1}{2}g(t)\|\Phi_x\|_2^2 - \frac{1}{2}(g' \circ \Psi_x)(t) \leq 0.$$

An integration over $(0, t)$ yields, for any $t > 0$,

$$\rho_1\|\Phi_t\|_2^2 + K\|\Phi_x + \Psi\|_2^2 + \rho_2\|\Psi_t\|_2^2 + b\|\Psi_x\|_2^2 + (g \circ \Psi_x)(t) \leq 0,$$

thus,

$$\varphi = \tilde{\varphi} = \psi = \tilde{\psi}.$$

Hence, Problem (P_1) has a unique strong solution.

If $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in H_0^1(0, L) \times L^2(0, L)$, then it follows from the density of $(H^2(0, L) \cap H_0^1(0, L)) \times H_0^1(0, L)$ in $H_0^1(0, L) \times L^2(0, L)$ that Problem (P_1) has a unique weak solution. ∎

Now, we introduce the energy functional

$$E(t) := \frac{1}{2}\int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \left(b - \int_0^t g(s)ds\right)\psi_x^2 + K(\varphi_x + \psi)^2\right)dx + \frac{1}{2}(g \circ \psi_x)(t), \tag{2.9}$$

where, for any $v \in L^2_{loc}(\mathbb{R}_+; L^2(0, L))$,

$$(g \circ v)(t) := \int_0^L \int_0^t g(t - s)(v(t) - v(s))^2 ds dx.$$

Lemma 2.2 ([33]) *Let (φ, ψ) be the solution of (P_1). Then,*

$$E'(t) = -\frac{1}{2}g(t)\int_0^L \psi_x^2 dx + \frac{1}{2}(g' \circ \psi_x)(t) \leq \frac{1}{2}(g' \circ \psi_x)(t) \leq 0, \quad \forall t \geq 0. \tag{2.10}$$

As in Jin *et al*. [79], we set, for any $0 < \alpha < 1$,

$$C_\alpha := \int_0^\infty \frac{g^2(s)}{\alpha g(s) - g'(s)} ds \qquad \text{and} \qquad h_\alpha(t) := \alpha g(t) - g'(t).$$

Lemma 2.3 ([79]) *Assume that conditions (**A.1**) holds. Then for any $v \in L^2_{loc}(\mathbb{R}_+; L^2(0, L))$, we have*

$$\int_0^L \left(\int_0^t g(t-s)(v(t) - v(s))ds\right)^2 dx \leq C_\alpha (h_\alpha \circ v)(t), \qquad \forall\, t \geq 0. \tag{2.11}$$

2.2 Technical Lemmas

In this section, we state and prove some lemmas needed to establish our main results.

Lemma 2.4 *Assume that (**A.1**) and (**A.2**) hold. Then, the functional F defined by*

$$F(t) := -\rho_2 \int_0^L \psi_t \int_0^t g(t-s)(\psi(t) - \psi(s))dsdx$$

satisfies, along the solution of (P_1) and for any $0 < \delta < 1$, the estimates

$$F'(t) \quad \leq \quad -\rho_2 \left(\int_0^t g(s)ds - \delta\right) \int_0^L \psi_t^2 dx + \delta K \int_0^L (\varphi_x + \psi)^2 dx$$

$$+\delta\int_0^L \psi_x^2 dx + \frac{c}{\delta}(C_\alpha+1)(h_\alpha\circ\psi_x)(t). \tag{2.12}$$

Proof. Differentiating F and using equations in (P_1), we get

$$\begin{aligned}
F'(t) \;=\; & \left(b-\int_0^t g(s)ds\right)\int_0^L \psi_x \int_0^t g(t-s)(\psi_x(t)-\psi_x(s))dsdx \\
& +K\int_0^L(\varphi_x+\psi)\int_0^t g(t-s)(\psi(t)-\psi(s))dsdx \\
& +\int_0^L\left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds\right)^2 dx-\rho_2\left(\int_0^t g(s)ds\right)\int_0^L \psi_t^2 dx \\
& -\rho_2\int_0^L \psi_t\int_0^t g'(t-s)(\psi(t)-\psi(s))dsdx.
\end{aligned}$$

Now, we estimate the terms in the right-hand side of the above equation.

Using Young's inequality and Lemma 2.3, we obtain, for any $0<\delta<1$,

$$\begin{aligned}
& \left(b-\int_0^t g(s)ds\right)\int_0^L \psi_x \int_0^t g(t-s)(\psi_x(t)-\psi_x(s))dsdx \\
& \leq \delta\int_0^L \psi_x^2 + \frac{1}{4\delta}\left(b-\int_0^t g(s)ds\right)^2\int_0^L\left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds\right)^2 dx \\
& \leq \delta\int_0^L \psi_x^2 + \frac{c}{\delta}C_\alpha(h_\alpha\circ\psi_x)(t).
\end{aligned}$$

Also, we have

$$\begin{aligned}
& K\int_0^L(\varphi_x+\psi)\int_0^t g(t-s)(\psi(t)-\psi(s))dsdx \\
& \quad\leq \delta K\int_0^L(\varphi_x+\psi)^2dx+\frac{K}{4\delta}\int_0^L\left(\int_0^t g(t-s)(\psi(t)-\psi(s))ds\right)^2 dx \\
& \quad\leq \delta K\int_0^L(\varphi_x+\psi)^2dx+\frac{c}{\delta}C_\alpha(h_\alpha\circ\psi)(t) \\
& \quad\leq \delta K\int_0^L(\varphi_x+\psi)^2dx+\frac{c}{\delta}C_\alpha(h_\alpha\circ\psi_x)(t),
\end{aligned}$$

and, for $0<\delta<1$,

$$\int_0^L \left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds\right)^2 dx \le C_\alpha(h_\alpha\circ\psi_x)(t) \le \frac{c}{\delta}C_\alpha(h_\alpha\circ\psi_x)(t).$$

Exploiting Young's inequality and Lemma 2.3, we obtain, for any $0<\delta<1$,

$$\begin{aligned}
&-\rho_2\int_0^L \psi_t \int_0^t g'(t-s)(\psi(t)-\psi(s))dsdx \\
&= \rho_2\int_0^L \psi_t\int_0^t h_\alpha(t-s)(\psi(t)-\psi(s))dsdx \\
&\quad -\rho_2\int_0^L \psi_t\int_0^t \alpha g(t-s)(\psi(t)-\psi(s))dsdx \\
&\le \frac{\delta}{2}\rho_2\int_0^L \psi_t^2dx + \frac{\rho_2}{2\delta}\int_0^L\left(\int_0^t \sqrt{h_\alpha(t-s)}\sqrt{h_\alpha(t-s)}(\psi(t)-\psi(s))ds\right)^2 dx \\
&\quad +\frac{\delta}{2}\rho_2\int_0^L \psi_t^2dx + \frac{\rho_2}{2\delta}\alpha^2\int_0^L\left(\int_0^t g(t-s)(\psi(t)-\psi(s))ds\right)^2 dx \\
&\le \delta\rho_2\int_0^L \psi_t^2dx + \frac{\rho_2}{2\delta}\left(\int_0^t h_\alpha(s)ds\right)(h_\alpha\circ\psi)(t) + \frac{c}{\delta}C_\alpha(h_\alpha\circ\psi)(t) \\
&\le \delta\rho_2\int_0^L \psi_t^2dx + \frac{c}{\delta}(C_\alpha+1)(h_\alpha\circ\psi_x)(t).
\end{aligned}$$

A combination of all the above estimates gives the desired result. ∎

Lemma 2.5 *Under the conditions (**A.1**) and (**A.2**), the functional I_1 defined by*

$$I_1(t) := -\int_0^L (\rho_1\varphi\varphi_t + \rho_2\psi\psi_t)dx$$

satisfies, along the solution of (P_1), the estimate

$$I_1'(t) \le -\int_0^L(\rho_1\varphi_t^2+\rho_2\psi_t^2)dx + K\int_0^L(\varphi_x+\psi)^2dx + c\int_0^L \psi_x^2dx + cC_\alpha(h_\alpha\circ\psi_x)(t). \quad (2.13)$$

Proof. Using equations of (P_1) and repeating the above computations, we arrive at

$$\begin{aligned} I_1'(t) &= -\int_0^L (\rho_1\varphi_t^2 + \rho_2\psi_t^2)dx + K\int_0^L (\varphi_x+\psi)^2 dx \\ &+ \left(b - \int_0^t g(s)ds\right)\int_0^L \psi_x^2 dx + \int_0^L \psi_x \int_0^t g(t-s)(\psi_x(t)-\psi_x(s))dsdx \\ &\leq -\int_0^L (\rho_1\varphi_t^2 + \rho_2\psi_t^2)dx + K\int_0^L (\varphi_x+\psi)^2 dx + c\int_0^L \psi_x^2 + cC_\alpha(h_\alpha \circ \psi_x)(t). \end{aligned}$$

∎

Lemma 2.6 *Assume that (**A.1**) and (**A.2**) hold. Then, for any $0 < \varepsilon < 1$, the functional I_2 defined by*

$$I_2(t) := \rho_2 \int_0^L \psi_t(\varphi_x+\psi)dx + \frac{b\rho_1}{K}\int_0^L \varphi_t\psi_x dx - \frac{\rho_1}{K}\int_0^L \varphi_t \int_0^t g(t-s)\psi_x(s)dsdx$$

satisfies, along the solution of (P_1), the estimate

$$\begin{aligned} I_2'(t) &\leq \left[\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)\varphi_x\right]_{x=0}^{x=L} - K\int_0^L (\varphi_x+\psi)^2 dx \\ &+ c\varepsilon\rho_1 \int_0^L \varphi_t^2 dx + \rho_2\int_0^L \psi_t^2 dx + \frac{c}{\varepsilon}\int_0^L \psi_x^2 dx \\ &+ \frac{c}{\varepsilon}(C_\alpha+1)(h_\alpha\circ\psi_x)(t) + \left(\frac{b\rho_1}{K} - \rho_2\right)\int_0^L \varphi_t\psi_{xt}dx. \end{aligned} \tag{2.14}$$

Proof. Using the equations of (P_1), integrating by parts, applying Young's inequality and Lemma (2.3), then similar computations as in Lemma 2.4 yield

$$I_2'(t) = \left[\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)\varphi_x\right]_{x=0}^{x=L} - K\int_0^L (\varphi_x+\psi)^2 dx$$

$$
\begin{aligned}
&+\rho_2 \int_0^L \psi_t^2 dx - \frac{\rho_1}{K} g(t) \int_0^L \varphi_t \psi_x dx + \left(\frac{b\rho_1}{K} - \rho_2\right) \int_0^L \varphi_x \psi_{xt} dx \\
&+\frac{\rho_1}{K} \int_0^L \varphi_t \int_0^t g'(t-s)(\psi_x(t) - \psi_x(s)) ds dx + \left(\frac{b\rho_1}{K} - \rho_2\right) \int_0^L \varphi_x \psi_{xt} dx \\
\leq \;& \left[\left(b\psi_x - \int_0^t g(t-s)\psi_x(s) ds\right)\varphi_x\right]_{x=0}^{x=L} - K \int_0^L (\varphi_x + \psi)^2 dx \\
&+\rho_2 \int_0^L \psi_t^2 dx + \frac{g(0)\rho_1}{K} \varepsilon \int_0^L \varphi_t^2 dx + \frac{g(0)\rho_1}{4K\varepsilon} \int_0^L \psi_x^2 dx \\
&+\frac{\rho_1}{K} \varepsilon \int_0^L \varphi_t^2 dx + \frac{\rho_1}{4K\varepsilon} \int_0^L \left(\int_0^t g'(t-s)(\psi_x(t) - \psi_x(s)) ds\right)^2 dx \\
&+\left(\frac{b\rho_1}{K} - \rho_2\right) \int_0^L \varphi_x \psi_{xt} dx \\
\leq \;& \left[\left(b\psi_x - \int_0^t g(t-s)\psi_x(s) ds\right)\varphi_x\right]_{x=0}^{x=L} - K \int_0^L (\varphi_x + \psi)^2 dx \\
&+c\varepsilon\rho_1 \int_0^L \varphi_t^2 dx + \rho_2 \int_0^L \psi_t^2 dx + \frac{c}{\varepsilon} \int_0^L \psi_x^2 dx \\
&+\frac{c}{\varepsilon}(C_\alpha + 1)(h_\alpha \circ \psi_x)(t) + \left(\frac{b\rho_1}{K} - \rho_2\right) \int_0^L \varphi_t \psi_{xt} dx.
\end{aligned}
$$

$\blacksquare$

Lemma 2.7 *Assume that (**A.1**) and (**A.2**) hold. Let $m(x) = 2 - \frac{4}{L}x$, for $x \in [0, L]$. Then, for any $0 < \varepsilon < 1$, the functional I_3 defined by*

$$
I_3(t) := \frac{\rho_2}{4\varepsilon} \int_0^L m(x)\psi_t \left(b\psi_x - \int_0^t g(t-s)\psi_x(s) ds\right) dx + \varepsilon \frac{\rho_1}{K} \int_0^L m(x)\varphi_t \varphi_x dx
$$

satisfies, along the solution of (P_1), the estimate

$$
\begin{aligned}
I_3'(t) \;\leq\; & -\frac{1}{4}\left[\left(b\psi_x(L,t) - \int_0^t g(t-s)\psi_x(L,s) ds\right)^2 + \left(b\psi_x(0,t) - \int_0^t g(t-s)\psi_x(0,s) ds\right)^2\right] \\
&-\varepsilon\Big(\varphi_x^2(L,t) + \varphi_x^2(0,t)\Big) + \left(\frac{1}{4} + c\varepsilon\right) K \int_0^L (\varphi_x + \psi)^2 dx + c\varepsilon\rho_1 \int_0^L \varphi_t^2 dx \\
&+\frac{c}{\varepsilon}\rho_2 \int_0^L \psi_t^2 dx + \frac{c}{\varepsilon^2} \int_0^L \psi_x^2 dx + \frac{c}{\varepsilon^2}(C_\alpha + 1)(h_\alpha \circ \psi_x)(t). \qquad (2.15)
\end{aligned}
$$

Proof. Exploiting the equations of (P_1), Young's and Poicaré's inequalities, Lemma 2.3 and the relation

$$\varphi_x^2 \leq 2(\varphi_x + \psi)^2 + 2\psi^2,$$

we have

$$\begin{aligned}
I_3'(t) &= \frac{1}{4\varepsilon}\left[-\left(b\psi_x(L,t) - \int_0^t g(t-s)\psi_x(L,s)ds\right)^2\right. \\
&\quad -\left(b\psi_x(0,t) - \int_0^t g(t-s)\psi_x(0,s)ds\right)^2 \\
&\quad -\frac{1}{2}\int_0^L m'(x)\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2 dx \\
&\quad -K\int_0^L m(x)\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)(\varphi_x + \psi)dx \\
&\quad -\frac{b\rho_2}{2}\int_0^L m'(x)\psi_t^2 dx - \rho_2 g(t)\int_0^L m(x)\psi_x\psi_t dx \\
&\quad \left. +\rho_2\int_0^L m(x)\psi_t\int_0^t g'(t-s)(\psi_x(t) - \psi_x(s))dsdx\right] \\
&\quad +\varepsilon\left[-(\varphi_x^2(L,t) + \varphi_x^2(0,t)) + \int_0^L m(x)\varphi_x\psi_x dx\right. \\
&\quad \left. -\frac{1}{2}\int_0^L m'(x)\varphi_x^2 dx - \frac{\rho_1}{2K}\int_0^L m'(x)\varphi_t^2 dx\right] \\
&\leq -\frac{1}{4\varepsilon}\left[\left(b\psi_x(L,t) - \int_0^t g(t-s)\psi_x(L,s)ds\right)^2\right. \\
&\quad \left. +\left(b\psi_x(0,t) - \int_0^t g(t-s)\psi_x(0,s)ds\right)^2\right] \\
&\quad -\varepsilon\Big(\varphi_x^2(L,t) + \varphi_x^2(0,t)\Big) + \left(\frac{1}{4} + c\varepsilon\right)K\int_0^L (\varphi_x + \psi)^2 dx \\
&\quad +c\varepsilon\rho_1\int_0^L \varphi_t^2 dx + \frac{c}{\varepsilon}\rho_2\int_0^L \psi_t^2 dx \\
&\quad +\frac{c}{\varepsilon^2}\int_0^L \psi_x^2 dx + \frac{c}{\varepsilon^2}(C_\alpha + 1)(h_\alpha \circ \psi_x)(t).
\end{aligned}$$

∎

Lemma 2.8 *Assume that (**A.1**) and (**A.2**) hold, after fixing ε small enough, the functional I defined by*

$$I(t) := 3c\varepsilon I_1(t) + I_2(t) + I_3(t)$$

satisfies, along the solution of (P_1) and for some constant $c_1 > 0$, the estimate

$$\begin{aligned} I'(t) \leq & -\frac{K}{2}\int_0^L (\varphi_x + \psi)^2 dx - c_1\rho_1 \int_0^L \varphi_t^2 dx \\ & + c\rho_2 \int_0^L \psi_t^2 dx + c\int_0^L \psi_x^2 dx \\ & + c(C_\alpha + 1)(h_\alpha \circ \psi_x)(t) + \left(\frac{b\rho_1}{K} - \rho_2\right)\int_0^L \varphi_t \psi_{xt} dx. \end{aligned} \tag{2.16}$$

Proof. Using Lemmas 2.5, 2.6, 2.7 and the inequality

$$\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)\varphi_x \leq \frac{1}{4\varepsilon}\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2 + \varepsilon\varphi_x^2,$$

then choosing ε so that $4c\varepsilon - \frac{3}{4} \leq -\frac{1}{2}$. Once ε is fixed, we set $c_1 = c\varepsilon$ and obtain the required result. ∎

As in [33] and [35], we use the multiplier

$$w(x,t) = \frac{1}{L}\left(\int_0^L \psi(y,t)dy\right)x - \int_0^x \psi(y,t)dy$$

which satisfies, for some $c_2 > 0$,

$$\int_0^L w_x^2 dx \leq \int_0^L \psi^2 dx \qquad \text{and} \qquad \int_0^L w_t^2 dx \leq c_2 \int_0^L \psi_t^2 dx. \tag{2.17}$$

Lemma 2.9 *Assume that* (**A.1**) *and* (**A.2**) *hold. Then, the functional* J *defined by*

$$J(t) := \int_0^L (\rho_1 w\varphi_t + \rho_2 \psi\psi_t)dx$$

satisfies, along the solution of (P_1), *the estimate*

$$J'(t) \leq \varepsilon_0\rho_1 \int_0^L \varphi_t^2 dx + \frac{c}{\varepsilon_0}\rho_2 \int_0^L \psi_t^2 dx - \frac{l}{2}\int_0^L \psi_x^2 dx + \frac{C_\alpha}{2l}(h_\alpha \circ \psi_x)(t), \quad \forall\, \varepsilon_0 > 0. \tag{2.18}$$

Proof. Using Young's inequality, Lemma 2.3 and (2.17), we get

$$\begin{aligned}
J'(t) &= \rho_1 \int_0^L w_t\varphi_t dx + K\int_0^L w(\varphi_x + \psi)_x dx + \rho_2 \int_0^L \psi_t^2 dx \\
&\quad + \int_0^L \psi \left(b\psi_{xx} - K(\varphi_x + \psi) - \int_0^t g(t-s)\psi_{xx}(s)ds \right) dx \\
&= \rho_1 \int_0^L w_t\varphi_t dx + K\int_0^L (w_x^2 - \psi^2)dx + \rho_2 \int_0^L \psi_t^2 dx \\
&\quad - \left(b - \int_0^t g(s)ds \right)\int_0^L \psi_x^2 dx + \int_0^L \psi_x \int_0^t g(t-s)(\psi_x(s) - \psi_x(t))dsdx \\
&\leq \varepsilon_0\rho_1 \int_0^L \varphi_t^2 dx + \frac{c}{\varepsilon_0}\rho_2 \int_0^L w_t^2 dx + \rho_2 \int_0^L \psi_t^2 dx - l\int_0^L \psi_x^2 dx \\
&\quad + \frac{l}{2}\int_0^L \psi_x^2 dx + \frac{C_\alpha}{2l}(h_\alpha \circ \psi_x) \\
&\leq \varepsilon_0\rho_1 \int_0^L \varphi_t^2 dx + \frac{c}{\varepsilon_0}\rho_2 \int_0^L \psi_t^2 dx - \frac{l}{2}\int_0^L \psi_x^2 dx + \frac{C_\alpha}{2l}(h_\alpha \circ \psi_x)(t).
\end{aligned}$$

∎

As in [23], let us define the functional

$$J_1(t) := \int_0^L \int_0^t f(t-s)\psi_x^2(s)dsdx,$$

where $f(t) := \int_t^\infty g(s)ds$ and state a lemma with its proof for completeness.

Lemma 2.10 *Assume that* (**A.1**) *and* (**A.2**) *hold. Then, the functional* J_1 *satisfies, along the solution of* (P_1), *the estimate*

$$J_1'(t) \leq -\frac{1}{2}(g \circ \psi_x)(t) + 3(b-l)\int_0^L \psi_x^2 dx. \tag{2.19}$$

Proof. Noting that $f'(t) = -g(t)$, we get

$$\begin{aligned}
J_1'(t) &= f(0)\int_0^L \psi_x^2 dx - \int_0^L \int_0^t g(t-s)\psi_x^2(s)dsdx \\
&= f(0)\int_0^L \psi_x^2 dx - \int_0^L \int_0^t g(t-s)(\psi_x(s)-\psi_x(t))^2 dsdx \\
&\quad -2\int_0^L \psi_x \int_0^t g(t-s)(\psi_x(s)-\psi_x(t))dsdx - \left(\int_0^t g(s)ds\right)\int_0^L \psi_x^2 dx \\
&= -(g\circ\psi_x)(t) - 2\int_0^L \psi_x \int_0^t g(t-s)(\psi_x(s)-\psi_x(t))dsdx + f(t)\int_0^L \psi_x^2 dx.
\end{aligned}$$

Exploiting Young's inequality and the fact that $\int_0^t g(s)ds \leq b-l$, we obtain

$$\begin{aligned}
&-2\int_0^L \psi_x \int_0^t g(t-s)(\psi_x(s)-\psi_x(t))dsdx \\
&\quad \leq 2(b-l)\int_0^L \psi_x^2 dx + \frac{1}{2(b-l)}\left(\int_0^t g(s)ds\right)\int_0^L\int_0^t g(t-s)(\psi_x(s)-\psi_x(t))^2 dsdx \\
&\quad \leq 2(b-l)\int_0^L \psi_x^2 dx + \frac{1}{2}(g\circ\psi_x)(t).
\end{aligned}$$

Using the last estimate and the fact that $f(t) \leq f(0) = b-l$, we arrive at the desired result. ∎

Lemma 2.11 *Let $t_0 > 0$ be fixed. Then, the functional $\mathcal{L}$ defined by*

$$\mathcal{L}(t) := NE(t) + N_1 F(t) + I(t) + N_2 J(t)$$

satisfies, for a suitable choice of $N,\ N_1,\ N_2 \geq 1$,

$$\mathcal{L}(t) \sim E(t) \tag{2.20}$$

and the estimate

$$\begin{aligned}\mathcal{L}'(t) \ \leq \ & -\frac{K}{4}\int_0^L (\varphi_x + \psi)^2 dx - \frac{\rho_1}{4}\int_0^L \varphi_t^2 dx - \frac{\rho_2}{4}\int_0^L \psi_t^2 dx - 4(b-l)\int_0^L \psi_x^2 dx \\ & + \frac{1}{4}(g \circ \psi_x)(t) + \left(\frac{b\rho_1}{K} - \rho_2\right)\int_0^L \varphi_t \psi_{xt} dx, \qquad \forall\, t \geq t_0. \end{aligned} \tag{2.21}$$

Proof. Using Hölder's, Poincaré's and Young inequalities, we get, for some $\beta_0 > 0$,

$$|\mathcal{L}(t) - NE(t)| \leq N_1 |F(t)| + |I(t)| + N_2 |J(t)| \leq \beta_0 E(t), \qquad \forall\, t \geq 0,$$

this yields

$$(N - \beta_0)E(t) \leq \mathcal{L}(t) \leq (N + \beta_0)E(t), \qquad \forall\, t \geq 0,$$

by taking $N > \beta_0$, we obtain (2.20).

To prove (2.21), set

$$g_0 = \int_0^{t_0} g(s)ds \qquad \text{and} \qquad \delta = \frac{1}{4N_1}.$$

Combining (2.10), (2.12), (2.16), (2.18) and recalling that $g' = \alpha g - h$, we obtain, for all $t \geq t_0$,

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\frac{K}{4}\int_0^L (\varphi_x + \psi)^2 dx - (c_1 - N_2\varepsilon_0)\rho_1 \int_0^L \varphi_t^2 dx \\
& - \left[g_0 N_1 - \frac{1}{4} - c\left(1 + \frac{1}{\varepsilon_0}N_2\right)\right]\rho_2 \int_0^L \psi_t^2 dx \\
& - \left(\frac{l}{2}N_2 - \frac{1}{4} - c\right)\int_0^L \psi_x^2 dx + \frac{\alpha}{2}N(g \circ \psi_x)(t) \\
& - \left[\frac{1}{2}N - c(4N_1^2 + 1) - C_\alpha\left(\frac{1}{2l}N_2 + c + 4cN_1^2\right)\right](h_\alpha \circ \psi_x)(t) \\
& + \left(\frac{b\rho_1}{K} - \rho_2\right)\int_0^L \varphi_t \psi_{xt} dx.
\end{aligned}$$

We start by choosing N_2 large enough so that

$$\frac{l}{2}N_2 - \frac{1}{4} - c > 4(b - l),$$

then pick ε_0 so small that

$$c_1 - N_2\varepsilon_0 > \frac{1}{4}.$$

Next, we select N_1 so large that

$$g_0 N_1 - \frac{1}{4} + c(1 + \frac{1}{\varepsilon_0}N_2) > \frac{1}{4}.$$

As $\dfrac{\alpha g^2(s)}{\alpha g(s) - g'(s)} < g(s)$, it follows from the Lebesgue Dominated Convergence Theorem that

$$\lim_{\alpha \to 0^+} \alpha C_\alpha = \lim_{\alpha \to 0^+} \int_0^\infty \frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} ds = 0.$$

Consequently, there exists $0 < \alpha_0 < 1$ such that if $\alpha < \alpha_0$, then

$$\alpha C_\alpha < \frac{1}{8\left[\frac{1}{2l}N_2 + c(1+4N_1^2)\right]}.$$

Now, choose N large enough so that

$$N > \max\left\{4c(4N_1^2+1),\ \frac{1}{2\alpha_0}\right\}$$

and set

$$\alpha = \frac{1}{2N}.$$

So

$$\frac{1}{4}N - c(4N_1^2+1) > 0 \qquad \text{and} \qquad \alpha = \frac{1}{2N} < \alpha_0.$$

This gives

$$\begin{aligned} \frac{1}{2}N - c(4N_1^2+1) - C_\alpha\left[\frac{1}{2l}N_2 + c(1+4N_1^2)\right] \ &> \ \frac{1}{2}N - c(4N_1^2+1) - \frac{1}{8\alpha} \\ &= \ \frac{1}{4}N - c(4N_1^2+1) > 0. \end{aligned}$$

Hence, we arrive at the required estimate. ∎

2.3 A Decay Result for Equal Speeds of Wave Propagation

In this section, we state and prove a new general decay result in the case of equal speeds of wave propagation.

Theorem 2.2 *Let $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in H_0^1(0,L) \times L^2(0,L)$. Assume that the hypotheses*

*(**A.1**) and (**A.2**) hold and*

$$\frac{K}{\rho_1} = \frac{b}{\rho_2}. \tag{2.22}$$

Then, there exist two positive constants C and λ such that the energy functional associated to problem (P_1) satisfies the estimate, for any $t > t_0$,

$$E(t) \leq C \exp\left(-\lambda \int_{t_0}^{t} \xi(s)ds\right), \qquad \text{for} \quad p = 1, \tag{2.23}$$

and

$$E(t) \leq C \left(1 + \int_{t_0}^{t} \xi(s)ds\right)^{-\frac{1}{p-1}}, \qquad \text{for} \quad 1 < p < 2. \tag{2.24}$$

Proof. Exploiting (2.22), estimate (2.21) becomes, for some $m > 0$ and for any $t \geq t_0$,

$$\begin{aligned} \mathcal{L}'(t) &\leq -\frac{K}{4}\int_0^L (\varphi_x + \psi)^2 dx - \frac{\rho_1}{4}\int_0^L \varphi_t^2 dx - \frac{\rho_2}{4}\int_0^L \psi_t^2 dx \\ &\quad -4(b-l)\int_0^L \psi_x^2 dx + \frac{1}{4}(g \circ \psi_x)(t) \\ &\leq -mE(t) + c(g \circ \psi_x)(t). \end{aligned} \tag{2.25}$$

Case $\boldsymbol{p = 1}$. Multiplying (2.25) by $\xi(t)$, then exploiting (**A.2**) and (2.10) we get

$$\xi(t)\mathcal{L}'(t) \leq -m\xi(t)E(t) + c\xi(t)(g \circ \psi_x)(t) \leq -m\xi(t)E(t) - cE'(t), \qquad \forall\, t \geq t_0.$$

Using the non-increasing property of ξ, we have $\xi\mathcal{L} + cE \sim E$ and

$$(\xi\mathcal{L} + cE)'(t) \leq -m\xi(t)E(t), \qquad \forall\, t \geq t_0.$$

A simple integration over (t_0, t) yields, for two positive constants C and λ,

$$E(t) \le C \exp\left(-\lambda \int_{t_0}^{t} \xi(s) ds\right), \qquad \forall\, t > t_0.$$

Case $1 < p < 2$. First, we use Lemmas 2.10 and 2.11 to conclude that

$$\mathscr{L}(t) := \mathcal{L}(t) + J_1(t)$$

is nonnegative and satisfies, for some $\beta > 0$ and for any $t \ge t_0$,

$$\begin{aligned} \mathscr{L}'(t) \;\le\; & -\frac{K}{4}\int_0^L (\varphi_x + \psi)^2 dx - \frac{\rho_1}{4}\int_0^L \varphi_t^2 dx - \frac{\rho_2}{4}\int_0^L \psi_t^2 dx \\ & -(b-l)\int_0^L \psi_x^2 dx - \frac{1}{4}(g \circ \psi_x)(t) \\ \le\; & -\beta E(t). \end{aligned}$$

It is worth noticing that this estimate yields

$$\int_0^{\infty} E(s) ds < +\infty \tag{2.26}$$

and

$$E(t) \le \frac{c}{t - t_0}, \qquad \forall\, t > t_0. \tag{2.27}$$

Now, we define a functional η by

$$\eta(t) := \int_0^t \|\psi_x(t) - \psi_x(t-s)\|_2^2\, ds,$$

where (3.14) implies that

$$\begin{aligned}\eta(t) &= \int_0^t \|\psi_x(t)-\psi_x(s)\|_2^2 ds \le 2\int_0^t \big(\|\psi_x(t)\|_2^2+\|\psi_x(s)\|_2^2\big)ds \\ &\le \frac{4}{l}\int_0^t \big(E(t)-E(s)\big)ds \le \frac{8}{l}\int_0^t E(s)ds < \infty \qquad \forall\, t\ge 0. \end{aligned} \tag{2.28}$$

We further assume that there exists $t_1>0$ such that $\eta(t_1)>0$, otherwise

$$(g\circ\psi_x)(t) \le g(0)\eta(t) = 0, \qquad \forall\, t\ge 0,$$

this together with (2.25) yield an exponential decay. Also, Without loss of generality, we assume $t_1 = t_0$, then it follows from Jensen's inequality, assumption (**A.2**), Lemma 2.2 and (2.28) that estimate (2.25) becomes

$$\begin{aligned}\mathcal{L} &\le -mE(t)+c\eta(t)\cdot\frac{1}{\eta(t)}\int_0^t g^{p\cdot\frac{1}{p}}(t-s)\|\psi_x(t)-\psi_x(s)\|_2^2 ds \\ &\le -mE(t)+c\eta(t)\left(\frac{1}{\eta(t)}\int_0^t g^{p}(t-s)\|\psi_x(t)-\psi_x(s)\|_2^2 ds\right)^{\frac{1}{p}} \\ &\le -mE(t)+c\eta^{1-1/p}(t)\left(-\int_0^t \frac{g'(t-s)}{\xi(t-s)}\|\psi_x(t)-\psi_x(s)\|_2^2 ds\right)^{\frac{1}{p}} \\ &\le -mE(t)+c\left(-\frac{1}{\xi(t)}\int_0^t g'(t-s)\|\psi_x(t)-\psi_x(s)\|_2^2 ds\right)^{\frac{1}{p}} \\ &\le -mE(t)+c\left(-\frac{E'(t)}{\xi(t)}\right)^{\frac{1}{p}}, \qquad \forall\, t\ge t_0.\end{aligned}$$

This last estimate together with Lemma 2.2 and Young's inequality yield, for any $\varepsilon>0$,

$$\big(E^{p-1}\mathcal{L}\big)'(t) = (p-1)E^{p-2}(t)E'(t)\mathcal{L}(t)+E^{p-1}(t)\mathcal{L}'(t)$$

$$
\begin{aligned}
&\leq \quad -mE^p(t) + cE^{p-1}(t)\left(-\frac{E'(t)}{\xi(t)}\right)^{\frac{1}{p}} \\
&\leq \quad -(m-\varepsilon)E^p(t) - \frac{c}{\varepsilon}\frac{E'(t)}{\xi(t)}, \qquad \forall\, t \geq t_0.
\end{aligned}
$$

By taking $\varepsilon < m$, we obtain, for some fixed $m_0 > 0$,

$$
\xi(t)\big(E^{p-1}\mathcal{L}\big)'(t) \leq -m_0\xi(t)E(t) - cE'(t), \qquad \forall\, t \geq t_0.
$$

Set $\mathcal{F} = \xi E^{p-1}\mathcal{L} + cE \sim E$, then the nonincreasing property of ξ gives, for any $t \geq t_0$,

$$
\mathcal{F}'(t) \leq -m_0\xi(t)E^p(t).
$$

We integrate over (t_0, t) to get

$$
E(t) \leq C\left(1 + \int_{t_0}^{t} \xi(s)ds\right)^{-\frac{1}{p-1}}, \qquad \forall\, t > t_0,
$$

where C is a positive constant. ∎

Remark 2.1 *It is evident that the decay rate deduced from estimate* (2.23) *is optimal in the sense that it agrees with the decay rate of g deduced from (**A.2**). For the details, see [23, Remark 2.3].*

Example 2.1

(1) Consider the relaxation function $g(t) = a\exp(-\alpha t)$, where a, α are positive

*constants and a is chosen so that hypothesis (**A.1**) is satisfied, then*

$$g'(t) = -\alpha g(t).$$

Therefore, it follows from (2.23) *that there exists $\lambda > 0$ such that*

$$E(t) \leq C \exp(-\alpha\lambda t), \qquad \forall\, t \geq 0.$$

*(2) Consider $g(t) = ae^{-(1+t)^{\nu}}$, for $0 < \nu < 1$ and a is chosen so that condition (**A.1**) is satisfied, then*

$$g'(t) = \xi(t)g(t) \qquad \text{with} \qquad \xi(t) = \nu(1+t)^{\nu-1}.$$

Estimate (2.23) *entails that*

$$E(t) \leq Ce^{-\lambda(1+t)^{\nu}}, \qquad \forall\, t \geq 0.$$

(3) Consider the following relaxation function, for $\nu > 1$,

$$g(t) = \frac{a}{(1+t)^{\nu}}$$

*and a is chosen so that hypothesis (**A.1**) remains valid. Then*

$$g'(t) = -bg^{p}(t) \qquad \text{with} \qquad p = \frac{\nu+1}{\nu},$$

where b is a fixed constant and it satisfies $1 < p < 2$. Then, we deduce from (2.24) *that*

$$E(t) \leq \frac{C}{(1+t)^{\nu}}, \qquad \forall\, t \geq 0.$$

For more examples, see [23], [35].

2.4 A Decay Result for Non-Equal Speeds of Wave Propagation

In this section, we give an estimate to the decay rate in the case of non-equal speeds of wave propagation. We start by differentiating both sides of the differential equations in (P_1) with respect to t and use the fact that

$$\frac{\partial}{\partial t}\left[\int_0^t g(t-s)\psi_{xx}(s)ds\right] = \int_0^t g(t-s)\psi_{xxt}(s)ds + g(t)\psi_{0xx},$$

to obtain the following system

$$\begin{cases} \rho_1\varphi_{ttt} - K(\varphi_{xt}+\psi_t)_x = 0, & \text{in } (0,L)\times(0,+\infty), \\ \rho_2\psi_{ttt} - b\psi_{xxt} + K(\varphi_{xt}+\psi_t) + \displaystyle\int_0^t g(t-s)\psi_{xxt}(s)ds + g(t)\psi_{0xx} = 0, & \text{in } (0,L)\times(0,+\infty). \end{cases}$$

First, we introduce the "second" energy functional

$$E_*(t) := \frac{1}{2}\int_0^L \left(\rho_1\varphi_{tt}^2 + \rho_2\psi_{tt}^2 + \left(b - \int_0^t g(s)ds\right)\psi_{xt}^2 + K(\varphi_{xt}+\psi_t)^2\right)dx + \frac{1}{2}(g\circ\psi_{xt})(t). \tag{2.29}$$

Then we have the following Lemma.

Lemma 2.12 ([77]) *Let (φ,ψ) be the strong solution of (P_1). Then, the second* *energy functional satisfies, for all $t \geq 0$,*

$$E'_*(t) = -\frac{1}{2}g(t)\int_0^L \psi_{xt}^2 dx + \frac{1}{2}(g' \circ \psi_{xt}) - g(t)\int_0^L \psi_{tt}\psi_{0xx}dx \tag{2.30}$$

and

$$E_*(t) \leq c\left(E_*(0) + \int_0^L \psi_{0xx}^2 dx\right). \tag{2.31}$$

Notice that E_* is not necessarily decreasing; but it is bounded.

Corollary 2.1 *Let (φ, ψ) be the strong solution of (P_1). Then, we have, for some positive constant c_1,*

$$0 \leq -(g' \circ \psi_{xt})(t) \leq c\big(-E'_*(t) + c_1 g(t)\big), \qquad \forall t \geq 0. \tag{2.32}$$

Proof. From equation (2.30) and inequality (2.31) we have, for any $t \geq 0$,

$$\begin{aligned}
0 \leq -(g' \circ \psi_{xt})(t) &= -2E'_*(t) - g(t)\int_0^L \psi_{xt}^2 dx - 2g(t)\int_0^L \psi_{tt}\psi_{0xx}dx \\
&\leq -2E'_*(t) - 2g(t)\int_0^L \psi_{tt}\psi_{0xx}dx \\
&\leq -2E'_*(t) + g(t)\int_0^L (\psi_{tt}^2 + \psi_{0xx}^2)dx \\
&\leq -2E'_*(t) + g(t)\left(\frac{2}{\rho_1}E_*(t) + \int_0^L \psi_{0xx}^2 dx\right) \\
&\leq c\Big(-E'_*(t) + c_1 g(t)\Big),
\end{aligned}$$

where c_1 is some fixed positive constant. ∎

Next, we have the following estimate for the last term in the right-hand side of (2.21).

Lemma 2.13 ([77]) *Let (φ,ψ) be the strong solution of (P_1). Then, for any $\varepsilon>0$, we have*

$$\left(\frac{\rho_1 b}{K}-\rho_2\right)\int_0^L \varphi_t\psi_{xt}dx \le \varepsilon E(t)+\frac{c}{\varepsilon}\Big((g\circ\psi_{xt})(t)-E'(t)+g(t)\Big),\quad \forall t\ge t_0. \quad (2.33)$$

Proof.

$$\begin{aligned}\left(\frac{\rho_1 k_2}{k_1}-\rho_2\right)\int_0^L \varphi_t\psi_{xt}dx \;&=\; \frac{\left(\frac{\rho_1 k_2}{k_2}-\rho_2\right)}{\int_0^t g(s)ds}\int_0^L \varphi_t\int_0^t g(t-s)(\psi_{xt}(t)-\psi_{xt}(s))dsdx\\ &\quad+\frac{\left(\frac{\rho_1 k_2}{k_1}-\rho_2\right)}{\int_0^t g(s)ds}\int_0^L \varphi_t\int_0^t g(t-s)\psi_{xt}(s)dsdx. \qquad (2.34)\end{aligned}$$

By observing that $\int_0^t g(s)ds \ge \int_0^{t_0} g(s)ds$, for all $t\ge t_0$ and exploiting Young's inequality and Lemma 2.3 (for ψ_{xt}), we get, for $\varepsilon>0$ and $t\ge t_0$,

$$\frac{\left(\frac{\rho_1 k_2}{k_1}-\rho_2\right)}{\int_0^t g(s)ds}\int_0^L \varphi_t\int_0^t g(t-s)(\psi_{xt}(t)-\psi_{xt}(s))dsdx \le \frac{\varepsilon}{4}\rho_1\int_0^L \varphi_t^2dx+\frac{c}{\varepsilon}(g\circ\psi_{xt}).$$

On the other hand, by integration by parts and using Lemma 2.3 (for $-g'$ and ψ_x) and the fact that E is non-increasing, we get

$$\begin{aligned}&\frac{\left(\frac{\rho_1 k_2}{k_1}-\rho_2\right)}{\int_0^t g(s)ds}\int_0^L \varphi_t\int_0^t g(t-s)\psi_{xt}(s)dsdx\\ =\;& \frac{\left(\frac{\rho_1 k_2}{k_1}-\rho_2\right)}{\int_0^t g(s)ds}\int_0^L \varphi_t\left(g(0)\psi_x-g(t)\psi_{0x}+\int_0^t g'(t-s)\psi_x(s)ds\right)dx\\ =\;& \frac{\left(\frac{\rho_1 k_2}{k_1}-\rho_2\right)}{\int_0^t g(s)ds}\int_0^L \varphi_t\left(g(t)(\psi_x-\psi_{0x})-\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))ds\right)dx\\ \le\;& \frac{\varepsilon}{4}\rho_1\int_0^L \varphi_t^2dx+\frac{c}{\varepsilon}g(t)\int_0^L(\psi_x^2+\psi_{0x}^2)dx-\frac{c}{\varepsilon}g'\circ\psi_x\end{aligned}$$

$$\leq \frac{\varepsilon}{4}\rho_1 \int_0^L \varphi_t^2 dx + \frac{c}{\varepsilon}E(0)g(t) - \frac{c}{\varepsilon}g' \circ \psi_x.$$

Inserting the last two inequalities in (2.34), we get (2.33). ∎

Now, we state and prove a general decay result in the case of non-equal speeds of wave propagation.

Theorem 2.3 *Let* $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in \big(H^2(0,L) \cap H_0^1(0,L)\big) \times H_0^1(0,L)$. *Assume that* (**A.1**) *–* (**A.2**) *hold and relation* (1.10) *is not satisfied, that is,*

$$\frac{\rho_1}{K} \neq \frac{\rho_2}{b}.$$

Then, for ant $t_0 > 0$ *there exists a positive constant* C *independent of* t *but may depend on the initial data such that the energy functional associated to problem* (P_1) *satisfies the estimate*

$$E(t) \leq Ct^{1-1/p} \left(\int_{t_0}^t \xi(s) ds \right)^{-\frac{1}{p}}, \qquad \forall\, t > t_0. \tag{2.35}$$

Proof. Using Lemma 2.13 in estimate (2.21), we have, for some $m > 0$,

$$\begin{aligned}
\mathcal{L}'(t) &\leq -mE(t) + c(g \circ \psi_x)(t) + \left(\frac{\rho_1 b}{K} - \rho_2 \right) \int_0^L \varphi_t \psi_{xt} dx \\
&\leq -(m - \varepsilon)E(t) + c(g \circ \psi_x)(t) + \frac{c}{\varepsilon}\big(g \circ \psi_{xt}(t) - E'(t) + g(t)\big), \quad \forall\, t \geq t_0.
\end{aligned}$$

After fixing ε small enough, we arrive at

$$\mathcal{L}'(t) \leq -m_1 E(t) + c\big(g \circ \psi_x + g \circ \psi_{xt}\big)(t) - cE'(t) + cg(t), \quad \forall\, t \geq t_0,$$

where m_1 is a fixed positive constant. By setting $\mathcal{F} := \mathcal{L} + cE \sim E$, we obatain, for any $t \geq t_0$,

$$\mathcal{F}'(t) \leq -m_1 E(t) + c\big(g \circ \psi_x + g \circ \psi_{xt}\big)(t) + cg(t). \tag{2.36}$$

Case $\boldsymbol{p = 1}$. Multiplying both sides of estimate (2.36) by $\xi(t)$, then using hypothesis ($\mathbf{A}.\mathbf{2}$), Lemma 2.2 and Corollary 2.1 we get, for any $t \geq t_0$,

$$\begin{aligned}
\xi(t)\mathcal{F}'(t) &\leq -m_1\xi(t)E(t) + c\xi(t)\big(g \circ \psi_x + g \circ \psi_{xt}\big)(t) + c\xi(t)g(t) \\
&\leq -m_1\xi(t)E(t) - cE'(t) + c\big(-E_*'(t) + c_1 g(t)\big) + c\xi(0)g(t).
\end{aligned}$$

From the non-increasing property of ξ, we have, for some fixed positive constant c_2,

$$\big(\xi\mathcal{F} + cE + cE_*\big)'(t) \leq -m_1\xi(t)E(t) + c_2 g(t), \quad \forall\, t \geq t_0,$$

which implies

$$m_1\xi(t)E(t) \leq -\big(\xi\mathcal{F} + cE + cE_*\big)'(t) + c_2 g(t), \quad \forall\, t \geq t_0.$$

An integration over (t_0, t), exploitation of the non-increasing property of E and estimate (2.31) yield, for any $t > t_0$,

$$\begin{aligned}
m_1 E(t)\int_{t_0}^{t}\xi(s)ds &\leq -\big(\xi\mathcal{F} + cE + cE_*\big)(t) + \big(\xi\mathcal{F} + cE + cE_*\big)(t_0) + c_2\int_{t_0}^{t} g(s)ds \\
&\leq \big(\xi\mathcal{F} + cE + cE_*\big)(0) + c\int_0^L \psi_{0xx}^2 dx + c_2(b - l).
\end{aligned}$$

Thus, we have, for some fixed positive constant C,

$$E(t) \leq \left(\int_{t_0}^{t} \xi(s) ds \right)^{-1}, \qquad \forall\, t > t_0.$$

Case $1 < p < 2$. First, we use estimates (2.9), (2.10) and (2.32) to observe that, for any $t \geq t_0$, the functional η defined by

$$\eta(t) := \frac{1}{t} \int_0^t \left(\|\psi_x(t) - \psi_x(t-s)\|_2^2 + \|\psi_{xt}(t) - \psi_{xt}(t-s)\|_2^2 \right) ds, \quad \forall\, t > 0,$$

satisfies

$$\begin{aligned}
\eta(t) &= \frac{1}{t} \int_0^t \left(\|\psi_x(t) - \psi_x(s)\|_2^2 + \|\psi_{xt}(t) - \psi_{xt}(s)\|_2^2 \right) ds \\
&\leq \frac{4}{lt} \int_0^t \big(E(t) + E(t-s) + E_*(t) + E_*(t-s) \big) ds \\
&\leq \frac{8}{lt} \int_0^t \left[E(0) + c \left(E_*(0) + \int_0^L \psi_{0xx}^2 dx \right) \right] ds \\
&\leq \frac{8}{l} \left[E(0) + c \left(E_*(0) + \int_0^L \psi_{0xx}^2 dx \right) \right] < +\infty, \quad \forall\, t > 0. \qquad (2.37)
\end{aligned}$$

There exists $t_1 > 0$ such that $\eta(t) \geq t_1$, otherwise $\eta(t) = 0$ for any $t \geq 0$, which in turn yields $(g \circ \psi_x)(t) = (g \circ \psi_{xt})(t) = 0$ for any $t \geq 0$. Thus, we have, from (2.36),

$$E(t) \leq C \left(\int_{t_0}^{t} \xi(s) ds \right)^{-1}, \qquad \forall\, t > t_0.$$

Without loss of generality, we assume $t_1 = t_0$, then from (**A.2**), (2.10), (2.32), (2.37) and Jensen's inequality we obtain

$$
\begin{aligned}
(g \circ \psi_x + &g \circ \psi_{xt})(t) \\
&= t\eta(t) \cdot \frac{1}{\eta(t)} \int_0^t g^{p \cdot \frac{1}{p}}(t-s) \cdot \frac{1}{t} \big[\|\psi_x(t) - \psi_x(s)\|_2^2 + \|\psi_{xt}(t) - \psi_{xt}(s)\|_2^2 \big] ds \\
&\le t\eta(t) \left(\frac{1}{t\eta(t)} \int_0^t g^p(t-s) \big[\|\psi_x(t) - \psi_x(s)\|_2^2 + \|\psi_{xt}(t) - \psi_{xt}(s)\|_2^2 \big] ds \right)^{1/p} \\
&\le t\eta^{1-1/p}(t) \left(-\frac{1}{t} \int_0^t \frac{g'(t-s)}{\xi(t-s)} \big[\|\psi_x(t) - \psi_x(s)\|_2^2 + \|\psi_{xt}(t) - \psi_{xt}(s)\|_2^2 \big] ds \right)^{1/p} \\
&\le ct \left(-\frac{1}{t\xi(t)} \int_0^t g'(t-s) \big[\|\psi_x(t) - \psi_x(s)\|_2^2 + \|\psi_{xt}(t) - \psi_{xt}(s)\|_2^2 \big] ds \right)^{1/p} \\
&\le ct \left(\frac{1}{t\xi(t)} \big[-E'(t) + c\big(-E'_*(t) + c_1 g(t) \big) \big] \right)^{1/p}, \qquad \forall\, t \ge t_0.
\end{aligned}
$$

Inserting this estimate in (2.36), we obtain, for any $t \ge t_0$,

$$
\mathcal{F}'(t) \le -m_1 E(t) - ct \left(\frac{1}{t\xi(t)} \big[-E'(t) + c(-E'_*(t) + c_1 g(t) \big) \right)^{1/p} + cg(t) \qquad (2.38)
$$

Next, we define a functional $\mathcal{F}_1$ by

$$
\mathcal{F}_1(t) := \left(\frac{E(t)}{t} \right)^{p-1} \mathcal{F}(t), \quad \forall\, t \ge t_0.
$$

Then, estimate (2.38) together with (2.10) and Young's inequality give, for any $\varepsilon > 0$ and for all $t \ge t_0$,

$$
\begin{aligned}
\mathcal{F}_1'(t) &= (p-1) \left(\frac{E(t)}{t} \right)^{p-2} \left(-\frac{E(t)}{t^2} + \frac{E'(t)}{t} \right) \mathcal{F}_1(t) + \left(\frac{E(t)}{t} \right)^{p-1} \mathcal{F}_1'(t) \\
&\le -m_1 E(t) \left(\frac{E(t)}{t} \right)^{p-1} + c \left(\frac{E(t)}{t} \right)^{p-1} g(t)
\end{aligned}
$$

$$
\begin{aligned}
&+ct\left(\frac{E(t)}{t}\right)^{p-1}\left(\frac{1}{t\xi(t)}\big[-E'(t)+c\big(-E'_*(t)+c_1g(t)\big)\big]\right)^{1/p}\\
\leq\ & -m_1t\left(\frac{E(t)}{t}\right)^{p}+c\left(\frac{E(0)}{t_0}\right)^{p-1}g(t)\\
&+ct\left(\frac{E(t)}{t}\right)^{p-1}\left(\frac{1}{t\xi(t)}\big[-E'(t)+c\big(-E'_*(t)+c_1g(t)\big)\big]\right)^{1/p}\\
\leq\ & -(m_1-\varepsilon)t\left(\frac{E(t)}{t}\right)^{p}+\frac{c}{\varepsilon\xi(t)}\big(-E'(t)-cE'_*(t)+c_2g(t)\big)+cg(t),
\end{aligned}
$$

where $c_2>0$ is a constant. We pick ε so that $m_1-\varepsilon>0$ and multiply the above inequality by $\xi(t)$, we get, for some c_3, $m_2>0$,

$$
\xi(t)\mathcal{F}'_1(t)\leq -m_2t\xi(t)\left(\frac{E(t)}{t}\right)^{p}-cE'(t)-cE'_*(t)+c_3g(t),\qquad \forall\, t\geq t_0.
$$

Let $\mathcal{F}_2=\xi\mathcal{F}_1+cE+cE_*\geq cE+cE'_*$, the nonincreasing property of ξ implies

$$
\mathcal{F}'_2(t)\leq -m_2t\xi(t)\left(\frac{E(t)}{t}\right)^{p}+cg(t),\qquad \forall\, t\geq t_0.
$$

Since the map $t\mapsto t\left(\frac{E(t)}{t}\right)^{p}$ is nonincreasing, we have, for ant $t\geq t_0$,

$$
\begin{aligned}
m_2t\left(\frac{E(t)}{t}\right)^{p}\int_{t_0}^{t}\xi(s)ds\ &\leq\ m_2\int_{t_0}^{t}s\left(\frac{E(s)}{s}\right)^{p}\xi(s)ds\leq -\int_{t_0}^{t}\big(\mathcal{F}_2(s)+c_3g(s)\big)ds\\
&\leq\ -\mathcal{F}_2(t)+\mathcal{F}_2(t_0)+c_3(b-l)\leq \mathcal{F}_2(t_0)+c_3(b-l).
\end{aligned}
$$

This last estimate implies that

$$
E(t)\leq Ct^{1-1/p}\left(\int_{t_0}^{t}\xi(s)ds\right)^{-1/p},\qquad \forall\, t>t_0.
$$

This completes the proof of Theorem 2.3. ∎

Example 2.2

(1) Consider the relaxation function $g(t) = a\exp(-\alpha t)$, *where* a, α *are positive constants and* a *is chosen so that hypothesis (*$\mathbf{A.1}$*) is satisfied, then*

$$g'(t) = -\alpha g(t) \qquad \forall\, t \geq 0.$$

Therefore, estimate (2.35) *implies that*

$$E(t) \leq \frac{c}{t - t_0}, \qquad \forall\, t > t_0.$$

(2) Consider $g(t) = ae^{-(1+t)^{\nu}}$, *for* $0 < \nu < 1$ *and* a *is chosen so that condition (*$\mathbf{A.1}$*) is satisfied, then*

$$g'(t) = -\xi(t) g(t) \qquad \forall\, t \geq 0, \qquad \text{where} \quad \xi(t) = \nu(1+t)^{\nu-1}.$$

Therefore, estimate (2.35) *entails that*

$$E(t) \leq \frac{c}{(1+t)^{\nu}}, \qquad \text{for} \quad t \quad \text{large} \quad \text{enough}.$$

(3) Consider the following relaxation function, for $\nu > 1$,

$$g(t) = \frac{a}{(1+t)^{\nu}}$$

and a is chosen so that hypothesis ($\mathbf{A.1}$) remains valid. Then

$$g'(t) = -bg^p(t), \qquad \forall\, t \geq 0,$$

where b is a fixed constant and $p = \frac{1+\nu}{\nu}$ satisfying $1 < p < 2$. Then, from (2.35) *that there exists $t_0 > 0$ such that*

$$E(t) \leq \frac{c}{(1+t)^{(2-p)/p}} = \frac{c}{(1+t)^{(\nu-1)/(\nu+1)}}, \qquad \forall\, t > t_0.$$

CHAPTER 3

GENERALIZED DECAY IN A VISCOELASTIC-TYPE BRESSE SYSTEM

In this chapter we will discuss the decay property of the following finite memory-type Bresse system:

$$
\begin{cases}
\rho_1\varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - lk_3(w_x - l\varphi) = 0, & \text{in } (0,L)\times(0,+\infty),\\
\rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + lw) + \displaystyle\int_0^t g(t-s)\psi_{xx}(s)ds = 0, & \text{in } (0,L)\times(0,+\infty),\\
\rho_1 w_{tt} - k_3(w_x - l\varphi)_x + lk_1(\varphi_x + \psi + lw) = 0, & \text{in } (0,L)\times(0,+\infty),\\
\varphi(0,t) = \varphi(L,t) = \psi_x(0,t) = \psi_x(L,t) = w_x(0,t) = w_x(L,t) = 0, & \text{for } t \geq 0,\\
\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), & \text{for } x \in (0,L),\\
\psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), & \text{for } x \in (0,L),\\
w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), & \text{for } x \in (0,L),
\end{cases}
\tag{P_2}
$$

where $l,\ k_1,\ k_2,\ k_3,\ \rho_1,\ \rho_2$ are positive constants, $\varphi_0,\ \varphi_1,\ \psi_0,\ \psi_1,\ w_0,\ w_1$ are given data and g is a relaxation function. This chapter is organized as follows: in Section 3.1, we state some preliminary results. In Section 3.2, we state and prove some technical lemmas. The statement and proof of our main results are given in Sections 3.3 and 3.4.

3.1 Preliminaries

In this section, we present some useful lemmas and state the well-posedness theorem.

Now, integrating both sides of the second and third equations in (P_2) over $(0, L)$ and using the boundary conditions, we get

$$\frac{d^2}{dt^2}\int_0^L \psi(x,t)dx + \frac{k_1}{\rho_2}\int_0^L \psi(x,t)dx + \frac{lk_1}{\rho_2}\int_0^L w(x,t)dx = 0 \quad \forall t \geq 0$$

and

$$\frac{d^2}{dt^2}\int_0^L w(x,t)dx + \frac{l^2k_1}{\rho_1}\int_0^L w(x,t)dx + \frac{lk_1}{\rho_1}\int_0^L \psi(x,t)dx = 0 \quad \forall t \geq 0.$$

Solving these ODEs simultaneously yields

$$\int_0^L \psi(x,t)dx = a_1\cos(a_0 t) + a_2\sin(a_0 t) + a_3 t + a_4$$

and

$$\int_0^L w(x,t)dx = \frac{a_1}{l}\left(\frac{\rho_2 a_0^2}{k_1} - 1\right)\cos(a_0 t) + \frac{a_2}{l}\left(\frac{\rho_2 a_0^2}{k_1} - 1\right)\sin(a_0 t) - \frac{a_3}{l}t - \frac{a_4}{l},$$

where

$$\begin{cases} a_0 = \sqrt{\dfrac{k_1}{\rho_2} + \dfrac{l^2 k_1}{\rho_1}} \\[2em] a_1 = \dfrac{k_1}{\rho_2 a_0^2}\displaystyle\int_0^L \psi_0(x)dx + \dfrac{lk_1}{\rho_2 a_0^2}\int_0^L w_0(x)dx, \\[2em] a_2 = \dfrac{k_1}{\rho_2 a_0^3}\displaystyle\int_0^L \psi_1(x)dx + \dfrac{lk_1}{\rho_2 a_0^3}\int_0^L w_1(x)dx, \\[2em] a_3 = \left(1 - \dfrac{k_1}{\rho_2 a_0^2}\right)\displaystyle\int_0^L \psi_1(x)dx - \dfrac{lk_1}{\rho_2 a_0^2}\int_0^L w_1(x)dx, \\[2em] a_4 = \left(1 - \dfrac{k_1}{\rho_2 a_0^2}\right)\displaystyle\int_0^L \psi_0(x)dx + \dfrac{lk_1}{\rho_2 a_0^2}\int_0^L w_0(x)dx. \end{cases}$$

Therefore, we perform the following change of variables

$$\widetilde{\psi} = \psi - \frac{1}{L}\big(a_1\cos(a_0 t) + a_2\sin(a_0 t) + a_3 t + a_4\big)$$

$$\widetilde{w} = w - \frac{1}{L}\left[\frac{a_1}{l}\left(\frac{\rho_2 a_0^2}{k_1} - 1\right)\cos(a_0 t) + \frac{a_2}{l}\left(\frac{\rho_2 a_0^2}{k_1} - 1\right)\sin(a_0 t) - \frac{a_3}{l}t - \frac{a_4}{l}\right]$$

to get

$$\int_0^L \widetilde{\psi}(x,t)dx = \int_0^L \widetilde{w}(x,t)dx = 0, \qquad \forall\, t \geq 0.$$

Furthermore, $(\varphi, \widetilde{\psi}, \widetilde{w})$ satisfies the equations and the boundary conditions in (P_2) with the initial data

$$\widetilde{\psi}_0 = \psi_0 - \frac{1}{L}(a_1 + a_4), \qquad \widetilde{\psi}_1 = \psi_1 - \frac{1}{L}(a_0 a_2 + a_3)$$

$$\widetilde{w}_0 = w_0 - \frac{1}{L}\left[\frac{a_1}{l}\left(\frac{\rho_2 a_0^2}{k_1} - 1\right) - \frac{a_4}{l}\right], \quad \widetilde{w}_1 = w_1 - \frac{1}{L}\left[\frac{a_2 a_0}{l}\left(\frac{\rho_2 a_0^2}{k_1} - 1\right) - \frac{a_3}{l}\right].$$

From now on, we work with $\widetilde{\psi},\ \widetilde{w}$ and, respectively, write $\psi,\ w$ for convenience. We also introduce the following spaces,

$$L^2_*(0,L) := \left\{ w \in L^2(0,L) : \int_0^L w(x)dx = 0 \right\}, \qquad H^1_*(0,L) := H^1(0,L) \cap L^2_*(0,L),$$

and

$$H^2_*(0,L) := \left\{ w \in H^2(0,L) : w_x(0) = w_x(L) = 0 \right\}.$$

Then, Poincaré's inequality is applicable to the elements of $H^1_*(0,L)$, that is,

$$\exists \;\; c_0 > 0 \quad \text{such} \quad \text{that} \qquad \int_0^L v^2 dx \le c_0 \int_0^L v_x^2 dx, \qquad \forall\, v \in H^1_0(0,L) \cup H^1_*(0,L).$$

Lemma 3.1 *For* $l < \dfrac{1}{2\sqrt{c_0}}$*, there exist* $\beta_1,\ \beta_2 > 0$ *such that*

$$\begin{aligned}
\beta_1\big(\|\varphi_x\|_2^2 + \|\psi_x\|_2^2 + \|w_x\|_2^2\big) \;&\le\; k_1\|\varphi_x + \psi + lw\|_2^2 + k_2\|\psi_x\|_2^2 + k_3\|w_x - l\varphi\|_2^2 \\
&\le\; \beta_2\big(\|\varphi_x\|_2^2 + \|\psi_x\|_2^2 + \|w_x\|_2^2\big),
\end{aligned}$$

for all $(\varphi,\,\psi,\,w\big)\in H_0^1(0,L)\times H_*^1(0,L)\times H_*^1(0,L)$.

Proof. Using Chauchy-Schwarz inequality we obtain

$$\begin{aligned}
\|\varphi_x\|_2^2+\|\psi_x\|_2^2+\|w_x\|_2^2 &= \|\varphi_x+\psi+lw-(\psi+lw)\|_2^2+\|\psi_x\|_2^2+\|w_x+l\varphi-l\varphi\|_2^2 \\
&\le 2\|\varphi_x+\psi+lw\|_2^2+\|\psi_x\|_2^2+2\|w_x-l\varphi\|_2^2 \\
&\quad +4c_0l^2\big(\|\varphi_x\|_2^2+\|\psi_x\|_2^2+\|w_x\|_2^2\big).
\end{aligned}$$

Thus,

$$\beta_1\big(\|\varphi_x\|_2^2+\|\psi_x\|_2^2+\|w_x\|_2^2\big)\le k_1\|\varphi_x+\psi+lw\|_2^2+k_2\|\psi_x\|_2^2+k_3\|w_x-l\varphi\|_2^2,$$

where

$$\beta_1=\frac{1-4c_0l^2}{\max\{2/k_1,1/k_2,2/k_3\}}>0.$$

Similarly,

$$k_1\|\varphi_x+\psi+lw\|_2^2+k_2\|\psi_x\|_2^2+k_3\|w_x-l\varphi\|_2^2\le\beta_2\big(\|\varphi_x\|_2^2+\|\psi_x\|_2^2+\|w_x\|_2^2\big),$$

where

$$\beta_2=\max\{2(k_1+c_0l^2k_3),\ (2c_0k_1+k_2),\ 2(k_1c_0l^2+k_3)\}>0.$$

∎

For completeness, we state, without proof the global existence and regularity result which can be established by repeating the steps of the proof of Theorem 4.1.

Theorem 3.1 *Let* $(\varphi_0, \varphi_1) \in H_0^1(0,L) \times L^2(0,L)$ *and* (ψ_0, ψ_1), $(w_0, w_1) \in H_*^1(0,L) \times L_*^2(0,L)$ *be given. Assume that* g *satisfies hypothesis* $(A1)$*. Then, the problem* (P_2) *has a unique global (weak) solution*

$$\varphi \in C(\mathbb{R}_+; H_0^1(0,L)) \cap C^1(\mathbb{R}_+; L^2(0,L)), \quad \psi,\ w \in C(\mathbb{R}_+; H_*^1(0,L)) \cap C^1(\mathbb{R}_+; L_*^2(0,L)).$$

Moreover, if

$$(\varphi_0, \varphi_1) \in (H^2(0,L) \cap H_0^1(0,L)) \times H_0^1(0,L)$$

and

$$(\psi_0, \psi_1),\ (w_0, w_1) \in (H_*^2(0,L) \cap H_*^1(0,L)) \times H_*^1(0,L),$$

then

$$\varphi \in C(\mathbb{R}_+; H^2(0,L) \cap H_0^1(0,L)) \cap C^1(\mathbb{R}_+; H_0^1(0,L)) \cap C^2(\mathbb{R}_+; L^2(0,L)),$$

and

$$\psi,\ w \in C(\mathbb{R}_+; H_*^2(0,L) \cap H_*^1(0,L)) \cap C^1(\mathbb{R}_+; H_*^1(0,L)) \cap C^2(\mathbb{R}_+; L_*^2(0,L)).$$

Now, we introduce the energy functional

$$\begin{aligned} E(t) := \frac{1}{2} \int_0^L & \left[\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 w_t^2 + \left(k_2 - \int_0^t g(s)ds \right) \psi_x^2 \right. \\ & \left. + k_3 (w_x - l\varphi)^2 + k_1 (\varphi_x + \psi + lw)^2 \right] dx + \frac{1}{2} (g \circ \psi_x)(t), \quad \forall\, t \geq 0, \end{aligned} \tag{3.1}$$

where for any $v \in L^2_{loc}([0,+\infty); L^2(0,L))$,

$$(g \circ v)(t) := \int_0^L \int_0^t g(t-s)\big(v(t)-v(s)\big)^2 ds dx.$$

By multiplying the equations in (P_2) by φ_t, ψ_t, w_t, respectively, integrating over $(0,L)$ and exploiting the boundary conditions we have the following lemma.

Lemma 3.2 *Let (φ, ψ, w) be the weak solution of (P_2). Then,*

$$E'(t) = -\frac{1}{2}g(t)\int_0^L \psi_x^2 dx + \frac{1}{2}(g' \circ \psi_x)(t) \leq 0, \qquad \forall t \geq 0. \tag{3.2}$$

3.2 Technical Lemmas

In this section, we state and prove some lemmas needed to establish our main results. All the computations are done for regular solutions but they still hold for weak and strong solutions by a density argument.

Lemma 3.3 *Assume that conditions (**A.1**) and (**A.2**) hold. Then, for any $0 < \delta < 1$, the functional I_1 defined by*

$$I_1(t) := -\rho_2 \int_0^L \psi_t \int_0^t g(t-s)(\psi(t)-\psi(s)) ds dx$$

satisfies, along the solution of (P_2), the estimates

$$\begin{aligned} I_1'(t) \leq &-\rho_2 \left(\int_0^t g(s)ds - \delta\right)\int_0^L \psi_t^2 dx + \delta \int_0^L (\varphi_x + \psi + lw)^2 dx \\ &+ c\delta \int_0^L \psi_x^2 dx + \frac{c}{\delta}(C_\alpha + 1)(h_\alpha \circ \psi)(t). \end{aligned} \tag{3.3}$$

Proof. Differentiating I_1, using equations in (P_2) and integrating by parts, we get

$$\begin{aligned} I_1'(t) =& -\rho_2 \int_0^L \psi_t \int_0^t g'(t-s)(\psi(t)-\psi(s))dsdx - \rho_2 \left(\int_0^t g(s)ds\right) \int_0^L \psi_t^2 dx \\ &+ \left(k_2 - \int_0^t g(s)ds\right) \int_0^L \psi_x \int_0^t g(t-s)(\psi_x(t)-\psi_x(s))dsdx \\ &+ k_1 \int_0^L (\varphi_x + \psi + lw) \int_0^t g(t-s)(\psi(t)-\psi(s))dsdx \\ &+ \int_0^L \left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds\right)^2 dx. \end{aligned}$$

Next, we estimate the terms on the right-hand side of the above equation.

Using Young's inequality, Lemma 2.3 and Poicaré's inequality, we obtain, for any $0 < \delta < 1$,

$$\begin{aligned} &-\rho_2 \int_0^L \psi_t \int_0^t g'(t-s)(\psi(t)-\psi(s))dsdx \\ &= \rho_2 \int_0^L \psi_t \int_0^t h_\alpha(t-s)(\psi(t)-\psi(s))dsdx \\ &\quad - \rho_2 \int_0^L \psi_t \int_0^t \alpha g(t-s)(\psi(t)-\psi(s))dsdx \\ &\le \frac{\delta}{2}\rho_2 \int_0^L \psi_t^2 dx + \frac{\rho_2}{2\delta} \int_0^L \left(\int_0^t \sqrt{h_\alpha(t-s)}\sqrt{h_\alpha(t-s)}(\psi(t)-\psi(s))ds\right)^2 dx \\ &\quad + \frac{\delta}{2}\rho_2 \int_0^L \psi_t^2 dx + \frac{\rho_2}{2\delta}\alpha^2 \int_0^L \left(\int_0^t g(t-s)(\psi(t)-\psi(s))ds\right)^2 dx \\ &\le \delta\rho_2 \int_0^L \psi_t^2 dx + \frac{\rho_2}{2\delta}\left(\int_0^t h_\alpha(s)ds\right)(h_\alpha \circ \psi)(t) + \frac{c}{\delta} C_\alpha (h_\alpha \circ \psi)(t) \\ &\le \delta\rho_2 \int_0^L \psi_t^2 dx + \frac{c}{\delta}(C_\alpha + 1)(h_\alpha \circ \psi_x)(t), \end{aligned}$$

$$\left(k_2 - \int_0^t g(s)ds\right) \int_0^L \psi_x \int_0^t g(t-s)(\psi_x(t)-\psi_x(t))dxds \le \delta \int_0^L \psi_x^2 dx + \frac{c}{\delta} C_\alpha (h_\alpha \circ \psi_x)(t),$$

$$k_1 \int_0^L (\varphi_x + \psi + lw) \int_0^t g(t-s)(\psi(t) - \psi(s))dsdx$$
$$\leq k_1 \delta \int_0^L (\varphi_x + \psi + lw)^2 dx + \frac{c}{\delta} C_\alpha (h_\alpha \circ \psi_x)(t),$$

and

$$\int_0^L \left(\int_0^t g(t-s)(\psi_x(t) - \psi_x(s))ds \right)^2 dx \leq c C_\alpha (h_\alpha \circ \psi_x)(t) \leq \frac{c}{\delta} C_\alpha (h_\alpha \circ \psi_x)(t).$$

A combination of these estimates gives the desired result. ∎

Lemma 3.4 *Assume that the hypotheses (***A.1***) and (***A.2***) hold. Then, for any* ε_0, $\delta_1 > 0$, *the functional* I_2 *defined by*

$$I_2(t) := -\rho_1 k_3 \int_0^L (w_x - l\varphi) \int_0^x w_t(y,t)dydx - \rho_1 k_1 \int_0^L \varphi_t \int_0^x (\varphi_x + \psi + lw)(y,t)dydx$$

satisfies, along the solution of (P_2), *the estimate*

$$\begin{aligned} I_2'(t) \leq & k_1^2 \int_0^L (\varphi_x + \psi + lw)^2 dx - k_3^2 \int_0^L (w_x - l\varphi)^2 dx + \frac{c}{\varepsilon_0} \int_0^L \psi_t^2 dx \\ & + \left(\varepsilon_0 - \rho_1 k_1 + \frac{l\rho_1 |k_3 - k_1| \delta_1}{2} \right) \int_0^L \varphi_t^2 dx \\ & + \rho_1 \left(k_3 + \frac{c_0 l |k_3 - k_1|}{2\delta_1} \right) \int_0^L w_t^2 dx. \end{aligned} \tag{3.4}$$

Proof. Differentiation of I_2, using equations in (P_2) and integration by parts yield

$$\begin{aligned}
I_2' =&\rho_1 k_3 \int_0^L w_t^2 dx + l\rho_1 k_3 \int_0^L \varphi_t \int_0^x w_t(y,t)dydx \\
&- k_3^2 \int_0^L (w_x - l\varphi)^2 dx + k_1^2 \int_0^L (\varphi_x + \psi + lw)^2 dx \\
&- \rho_1 k_1 \int_0^L \varphi_t^2 dx - \rho_1 k_1 \int_0^L \varphi_t \int_0^x (\psi_t + lw_t)(y,t)dydx.
\end{aligned}$$

Using Young's inequality, we get, for any ε_0, $\delta_1 > 0$,

$$\begin{aligned}
I_2' \leq& k_1^2 \int_0^L (\varphi_x + \psi + lw)^2 dx - k_3^2 \int_0^L (w_x - l\varphi)^2 dx + \frac{c}{\varepsilon_0} \int_0^L \psi_t^2 dx \\
&+ \left(\varepsilon_0 - \rho_1 k_1 + \frac{l\rho_1 |k_3 - k_1| \delta_1}{2}\right) \int_0^L \varphi_t^2 dx + \rho_1 \left(k_3 + \frac{c_0 l |k_3 - k_1|}{2\delta_1}\right) \int_0^L w_t^2 dx.
\end{aligned}$$

∎

Lemma 3.5 *Under the conditions* (**A.1**) *and* (**A.2**), *the functional* I_3 *defined by*

$$I_3(t) := -\rho_1 \int_0^L (\varphi_x + \psi + lw) w_t dx - \frac{k_3 \rho_1}{k_1} \int_0^L (w_x - l\varphi)\varphi_t dx$$

satisfies, along the solution of (P_2) *and for any* $\varepsilon_0 > 0$, *the estimate*

$$\begin{aligned}
I_3'(t) \leq& lk_1 \int_0^L (\varphi_x + \psi + lw)^2 dx - \frac{lk_3^2}{k_1} \int_0^L (w_x - l\varphi)^2 dx + \frac{c}{\varepsilon_0} \int_0^L \psi_t^2 dx \\
&+ \frac{l\rho_1 k_3}{k_1} \int_0^L \varphi_t^2 dx + (\varepsilon_0 - l\rho_1) \int_0^L w_t^2 dx + \rho_1 \left(\frac{k_3}{k_1} - 1\right) \int_0^L \varphi_{xt} w_t dx.
\end{aligned} \tag{3.5}$$

Proof. Differentiating I_3, using equations in (P_2) and integrating by parts, we have

$$\begin{aligned} I_3' = &- \rho_1 \int_0^L \psi_t w_t dx - l\rho_1 \int_0^L w_t^2 dx + lk_1 \int_0^L (\varphi_x + \psi + lw)^2 dx \\ &+ \frac{l\rho_1 k_3}{k_1} \int_0^L \varphi_t^2 dx - \frac{lk_3^2}{k_1} \int_0^L (w_x - l\varphi)^2 dx + \rho_1 \left(\frac{k_3}{k_1} - 1 \right) \int_0^L \varphi_{xt} w_t dx. \end{aligned}$$

Use of Young's inequality for the first term in the right-hand side gives (3.5). ∎

Lemma 3.6 *Assume that conditions (**A.1**) and (**A.2**) hold. Then for any* $0 < \delta < 1$, *the functional* I_4 *defined by*

$$I_4(t) := - \int_0^L (\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t + \rho_1 w w_t) dx$$

satisfies, along the solution of (P_2), *the estimate*

$$\begin{aligned} I_4'(t) \leq &- \int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 w_t^2) dx + k_1 \int_0^L (\varphi_x + \psi + lw)^2 dx \\ &+ k_3 \int_0^L (w_x - l\varphi)^2 dx + \left(k_2 + \delta - \int_0^t g(s) ds \right) \int_0^L \psi_x^2 dx + \frac{c}{\delta} C_\alpha (h_\alpha \circ \psi_x). \end{aligned} \tag{3.6}$$

Proof. Differentiation of I_4, using equations of (P_2) gives

$$\begin{aligned} I_4'(t) = &- \int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 w_t^2) dx + k_1 \int_0^L (\varphi_x + \psi + lw)^2 dx \\ &+ k_3 \int_0^L (w_x - l\varphi)^2 dx + \left(k_2 - \int_0^t g(s) ds \right) \int_0^L \psi_x^2 dx \\ &- \int_0^L \psi_x \int_0^t g(t-s)(\psi_x(t) - \psi_x(s)) ds dx. \end{aligned}$$

Repeating the above computations yields the desired result. ∎

Lemma 3.7 *Assume that conditions (**A.1**) and (**A.2**) hold. Then for any* $0 < \delta < 1$

and $\delta_2 > 0$, the functional I_5 defined by

$$I_5(t) := -\rho_2 \int_0^L \psi_x \int_0^x \psi_t(y,t)dydx$$

satisfies, along the solution of (P_2), the estimate

$$\begin{aligned} I_5'(t) \leq \rho_2 \int_0^L \psi_t^2 dx + \left(\frac{k_1}{2\delta_2} + \int_0^t g(s)ds + \delta - k_2\right) \int_0^L \psi_x^2 dx \\ + \frac{c_0 k_1 \delta_2}{2} \int_0^L (\varphi_x + \psi + lw)^2 dx + \frac{c}{\delta} C_\alpha(h_\alpha \circ \psi_x). \end{aligned} \tag{3.7}$$

Proof. Using equations of (P_2) and repeating similar computations as above, we arrive at

$$\begin{aligned} I_5'(t) &= \rho_2 \int_0^L \psi_t^2 dx - k_2 \int_0^L \psi_x^2 dx + k_1 \int_0^L \psi_x \int_0^x (\varphi_x + \psi + lw)(y,t)dydx \\ &\quad + \int_0^L \psi_x \int_0^t g(t-s)\psi_x(s)dsdx \\ &\leq \rho_2 \int_0^L \psi_t^2 dx + \left(\frac{k_1}{2\delta_2} + \int_0^t g(s)ds + \delta - k_2\right) \int_0^L \psi_x^2 dx \\ &\quad + \frac{k_1\delta_2}{2} \int_0^L \left(\int_0^x (\varphi_x + \psi + lw)(y,t)dy\right)^2 dx + \frac{c}{\delta} C_\alpha(h_\alpha \circ \psi_x)(t). \end{aligned}$$

Poincaré's inequality for the third term yields (3.7). ∎

Lemma 3.8 *Assume that the hypotheses (**A.1**) and (**A.2**) hold. Then, for any ε_0, ε_1, $\varepsilon_2 > 0$ and $0 < \delta < 1$, the functional I_6 defined by*

$$I_6(t) := \rho_2 \int_0^L \psi_t(\varphi_x + \psi + lw)dx + \frac{b\rho_1}{k_1} \int_0^L \varphi_t \psi_x dx - \frac{\rho_1}{k_1} \int_0^L \varphi_t \int_0^t g(t-s)\psi_x(s)dsdx$$

satisfies, along the solution of (P_2)*, the estimate*

$$\begin{aligned} I_6'(t) \leq {} & \delta \int_0^L \varphi_t^2 dx + \varepsilon_0 \int_0^L w_t^2 dx + \frac{c}{\varepsilon_0} \int_0^L \psi_t^2 dx - k_1 \int_0^L (\varphi_x + \psi + lw)^2 dx \\ & + \left(\frac{lk_2k_3\varepsilon_1}{2k_1} + \frac{lk_3\varepsilon_2}{2k_1} \int_0^t g(s)ds + \delta \right) \int_0^L (w_x - l\varphi)^2 dx \\ & + \left(\frac{lk_2k_3}{2k_1\varepsilon_1} + \frac{lk_3}{2k_1\varepsilon_2} \int_0^t g(s)ds + \frac{c}{\delta} g(t) \right) \int_0^L \psi_x^2 dx \\ & + \frac{c}{\delta}(C_\alpha + 1)(h_\alpha \circ \psi_x)(t) + \left(\frac{k_2\rho_1}{k_1} - \rho_2 \right) \int_0^L \varphi_t \psi_{xt} dx. \end{aligned} \tag{3.8}$$

Proof. Use of equations of (P_2) and integration by parts lead to

$$\begin{aligned} I_6'(t) = {} & -k_1 \int_0^L (\varphi_x + \psi + lw)^2 dx + \rho_2 \int_0^L \psi_t^2 dx + l\rho_2 \int_0^L \psi_t w_t dx \\ & + \frac{lk_2k_3}{k_1} \int_0^L (w_x - l\varphi)\psi_x dx - \frac{lk_3}{k_1} \int_0^L (w_x - l\varphi) \int_0^t g(t-s)\psi_x(s) ds dx \\ & - \frac{\rho_1}{k_1} g(t) \int_0^L \varphi_t \psi_x dx + \frac{\rho_1}{k_1} \int_0^L \varphi_t \int_0^t g'(t-s)(\psi_x(t) - \psi_x(s)) ds dx \\ & + \left(\frac{k_2\rho_1}{k_1} - \rho_2 \right) \int_0^L \varphi_x \psi_{xt} dx. \end{aligned}$$

Now, we estimate the terms in the right-hand side of the above equation.

Exploiting Young's inequality, we get

$$l\rho_2 \int_0^L \psi_t w_t dx \leq \varepsilon_0 \int_0^L w_t^2 dx + \frac{c}{\varepsilon_0} \int_0^L \psi_t^2 dx, \qquad \forall\, \varepsilon_0 > 0.$$

Using Young's inequality and Lemma 2.3, we obtain, for any $\varepsilon_1,\ \varepsilon_2 > 0$ and $0 < \delta < 1$,

$$\begin{aligned}
\frac{lk_2k_3}{k_1}\int_0^L (w_x - l\varphi)\psi_x dx - \frac{lk_3}{k_1}\int_0^L (w_x - l\varphi)\int_0^t g(t-s)\psi_x(s)dsdx \\
= \frac{lk_3}{k_1}\left(k_2 - \int_0^t g(s)ds\right)\int_0^L (w_x - l\varphi)\psi_x dx \\
+ \frac{lk_3}{k_1}\int_0^L (w_x - l\varphi)\int_0^t g(t-s)(\psi_x(t) - \psi_x(s))dsdx \\
\leq \left(\frac{lk_2k_3\varepsilon_1}{2k_1} + \frac{lk_3\varepsilon_2}{2k_1}\int_0^t g(s)ds + \delta\right)\int_0^L (w_x - l\varphi)^2 dx \\
+ \left(\frac{lk_2k_3}{2k_1\varepsilon_1} + \frac{lk_3}{2k_1\varepsilon_2}\int_0^t g(s)ds\right)\int_0^L \psi_x^2 dx + \frac{c}{\delta}C_\alpha(h_\alpha \circ \psi_x)
\end{aligned}$$

and

$$\begin{aligned}
-\frac{\rho_1}{k_1}g(t)\int_0^L \varphi_t\psi_x dx + \frac{\rho_1}{k_1}\int_0^L \varphi_t\int_0^t g'(t-s)(\psi_x(t) - \psi_x(s))dsdx \\
\leq \delta\int_0^L \varphi_t^2 dx + \frac{c}{\delta}g(t)\int_0^L \psi_x^2 dx - \frac{c}{\delta}(C_\alpha + 1)(h_\alpha \circ \psi_x)(t).
\end{aligned}$$

A combination of these estimates gives the desired result. ∎

As in [23], let us define the functional

$$J(t) := \int_0^L \int_0^t f(t-s)\psi_x^2(s)dsdx,$$

where $f(t) := \displaystyle\int_t^\infty g(s)ds$, then we have the following lemma.

Lemma 3.9 *Assume that (**A.1**) and (**A.2**) hold. Then, the functional J satisfies, along the solution of (P_2), the estimate*

$$J'(t) \leq -\frac{1}{2}(g \circ \psi_x)(t) + 3g_0\int_0^L \psi_x^2 dx, \tag{3.9}$$

where $g_0 = \int_0^\infty g(s)ds.$

Proof. Noting that $f'(t) = -g(t)$, we get

$$\begin{aligned}
J'(t) &= f(0)\int_0^L \psi_x^2 dx - \int_0^L \int_0^t g(t-s)\psi_x^2(s) ds dx \\
&= f(0)\int_0^L \psi_x^2 dx - \int_0^L \int_0^t g(t-s)(\psi_x(s)-\psi_x(t))^2 ds dx \\
&\quad -2\int_0^L \psi_x \int_0^t g(t-s)(\psi_x(s)-\psi_x(t)) ds dx - \left(\int_0^t g(s)ds\right)\int_0^L \psi_x^2 dx \\
&= -(g\circ\psi_x)(t) - 2\int_0^L \psi_x \int_0^t g(t-s)(\psi_x(s)-\psi_x(t)) ds dx + f(t)\int_0^L \psi_x^2 dx.
\end{aligned}$$

Exploiting Young's inequality and the fact that $\int_0^t g(s)ds \le b-l$, we obtain

$$\begin{aligned}
&-2\int_0^L \psi_x \int_0^t g(t-s)(\psi_x(s)-\psi_x(t)) ds dx \\
&\quad \le 2(b-l)\int_0^L \psi_x^2 dx + \frac{1}{2(b-l)}\left(\int_0^t g(s)ds\right)\int_0^L \int_0^t g(t-s)(\psi_x(s)-\psi_x(t))^2 ds dx \\
&\quad \le 2(b-l)\int_0^L \psi_x^2 dx + \frac{1}{2}(g\circ\ psi_x)(t).
\end{aligned}$$

Using the last estimate and the fact that $f(t) \le f(0) = b-l$, we arrive at the desired result. ∎

Lemma 3.10 *Let $t_0 > 0$ be fixed. Then, The functional $\mathcal{L}$ defined by*

$$\mathcal{L}(t) := NE(t) + \sum_{j=1}^{6} N_j I_j(t)$$

satisfies, for a suitable choice of N, $N_j \ge 0$ for $j = 1, 2, \cdots, 6$ with $N_3 = N_6 = 1$, and, *for l small enough,*

$$\mathcal{L}(t) \sim E(t) \tag{3.10}$$

and

$$\begin{aligned}\mathcal{L}'(t) \leq & -mE(t) + \frac{m}{2}\left(1 + \frac{k_2 - g_0}{12g_0}\right)(g \circ \psi_x)(t) \\ & + \left(\frac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \varphi_t \psi_{xt} dx + \rho_1\left(\frac{k_3}{k_2} - 1\right)\int_0^L \varphi_{xt} w_t dx, \quad \forall\, t \geq t_0.\end{aligned} \tag{3.11}$$

Proof. Using Hölder's, Poincaré's and Young inequalities, and Lemma 3.1, we obtain, for some $\beta_0 > 0$,

$$|\mathcal{L}(t) - NE(t)| \leq \sum_{j=1}^{6} N_j |I_j(t)| \leq \beta_0 E(t), \qquad \forall\, t \geq 0,$$

this implies that

$$(N - \beta_0)E(t) \leq \mathcal{L}(t) \leq (N + \beta_0)E(t), \qquad \forall\, t \geq 0.$$

Therefore, choosing $N > \beta_0$ if needed, we get (3.10).

For the proof of (3.11), a combination of (3.2), (3.3)–(3.8), and recall that $g' = \alpha g - h_\alpha$, yield, for all $t \geq t_0$,

$$\begin{aligned}\mathcal{L}'(t) \leq & \left[-\rho_1(k_1N_2 + N_4) + \frac{l\rho_1|k_3 - k_1|\delta_1 N_2}{2} + \frac{l\rho_1 k_3}{k_1} + \varepsilon_0 N_2 + \delta\right]\int_0^L \varphi_t^2 dx \\ & + \left[-\rho_2\left(N_1\int_0^t g(s)ds + N_4 - N_5\right) + \rho_2\delta N_1 + \frac{c}{\varepsilon_0}(1 + N_2)\right]\int_0^L \psi_t^2 dx \\ & + \left[-l\rho_1 + \rho_1(k_3N_2 - N_4) + \frac{c_0 l\rho_1|k_3 - k_1|N_2}{2\delta_1} + \varepsilon_0\right]\int_0^L w_t^2 dx\end{aligned}$$

$$
\begin{aligned}
&+\left[(N_5-N_4)\int_0^t g(s)ds+k_2(N_4-N_5)+\frac{k_1N_5}{2\delta_2}+\frac{lk_2k_3}{2k_1\varepsilon_1}\right.\\
&\left.+\frac{lk_3}{2k_1\varepsilon_2}\int_0^t g(s)ds+\delta(cN_1+N_4+N_5)+\frac{c}{\delta}g(t)\right]\int_0^L \psi_x^2dx\\
&+\left[-\frac{lk_3^2}{k_1}-k_3(k_3N_2-N_4)+\frac{lk_2k_3\varepsilon_1}{2k_1}\right.\\
&\left.+\frac{lk_2k_3\varepsilon_2}{2k_1}\int_0^t g(s)ds+\delta\right]\int_0^L (w_x-l\varphi)^2dx\\
&+\left[-k_1\left(1-k_1N_2-l-N_4-\frac{c_0\delta_2N_5}{2}\right)+\delta N_1\right]\int_0^L(\varphi_x+\psi+lw)^2dx\\
&+\frac{c}{\delta}\Big[(C_\alpha+1)+N_1(C_\alpha+1)+N_4C_\alpha+N_5C_\alpha\Big](h\circ\psi_x)(t)+NE'(t)\\
&+\left(\frac{k_2\rho_1}{k_1}-\rho_2\right)\int_0^L\varphi_t\psi_{xt}dx+\rho_1\left(\frac{k_3}{k_1}-1\right)\int_0^L\varphi_{xt}w_tdx.
\end{aligned}
$$

By setting $\delta_1=1$, $N_4=k_3N_2$, $N_5=4k_3N_2$, $\delta_2=\frac{k_1}{k_2-g_0}$, $\varepsilon_1=\frac{k_3}{k_2}$, and $\varepsilon_2=\frac{k_3}{2g_0}$, we arrive at

$$
\begin{aligned}
\mathcal{L}'(t)\le&-\rho_1\left[(k_1+k_3)N_2-l\left(\frac{|k_3-k_1|}{2}N_2+\frac{k_3}{k_1}\right)\right]\int_0^L\varphi_t^2dx\\
&-\rho_2\left(N_1\int_0^t g(s)ds-3k_3N_2\right)\int_0^L\psi_t^2dx\\
&-l\rho_1\left(1-\frac{c_0|k_3-k_1|}{2}N_2\right)\int_0^L w_t^2dx\\
&-\left[(k_2-g_0)k_3N_2-\frac{l}{k_1}\left(\frac{k_2^2}{2}+g_0^2\right)\right]\int_0^L\psi_x^2dx-\frac{lk_3^2}{4k_1}\int_0^L(w_x-l\varphi)^2dx\\
&-k_1\left[1-\left(k_1+k_3+\frac{2c_0k_1k_3}{k_2-g_0}\right)N_2-l\right]\int_0^L(\varphi_x+\psi+lw)^2dx\\
&+(1+N_2)\varepsilon_0\int_0^L(\varphi_t^2+w_t^2)dx+\frac{c}{\varepsilon_0}(1+N_2)\int_0^L\psi_t^2dx+NE'(t)\\
&+\delta\int_0^L\Big(\varphi_t^2+\rho_2N_1\psi_t^2+c(N_1+5k_3N_2)\psi_x^2+N_1(\varphi_x+\psi+lw)^2\Big)dx\\
&+\frac{c}{\delta}\Big[(1+N_1)+C_\alpha(1+N_1+5k_3N_2)\Big](h\circ\psi_x)(t)+\frac{c}{\delta}g(t)\int_0^L\psi_x^2dx
\end{aligned}
$$

$$+\left(\frac{k_2\rho_1}{k_1}-\rho_2\right)\int_0^L \varphi_t\psi_{xt}dx+\rho_1\left(\frac{k_3}{k_1}-1\right)\int_0^L \varphi_{xt}w_t dx.$$

Now, we set $\varepsilon_0=\frac{l\rho_1}{2(1+N_2)}$, to get

$$\begin{aligned}\mathcal{L}'(t)\le &-\rho_1\left[(k_1+k_3)N_2-l\left(\frac{1}{2}+\frac{k_3}{k_1}+\frac{|k_3-k_1|}{2}N_2\right)\right]\int_0^L \varphi_t^2dx\\
&-\rho_2\left(N_1\int_0^t g(s)ds-3k_3N_2-\frac{c(1+N_2)^2}{l\rho_1\rho_2}\right)\int_0^L \psi_t^2dx\\
&-\frac{lk_3^2}{4k_1}\int_0^L (w_x-l\varphi)^2dx-\frac{l\rho_1}{2}\left(1-c_0|k_3-k_1|N_2\right)\int_0^L w_t^2dx\\
&-\left[(k_2-g_0)k_3N_2-\frac{l}{k_1}\left(\frac{k_2^2}{2}+g_0^2\right)\right]\int_0^L \psi_x^2dx+\delta c_{N_1,N_2}E(t)\\
&-k_1\left[1-\left(k_1+k_3+\frac{2c_0k_1k_3}{k_2-g_0}\right)N_2-l\right]\int_0^L(\varphi_x+\psi+lw)^2dx\\
&+\left[N-\frac{c}{\delta}\right]E'(t)+\frac{c}{\delta}\Big[(1+N_1)+C_\alpha(1+N_1+5k_3N_2)\Big](h\circ\psi_x)(t)\\
&+\left(\frac{k_2\rho_1}{k_1}-\rho_2\right)\int_0^L \varphi_t\psi_{xt}dx+\rho_1\left(\frac{k_3}{k_1}-1\right)\int_0^L \varphi_{xt}w_tdx.\end{aligned}$$

Choose N_2 so small that

$$1-c_0|k_3-k_1|N_2>0 \qquad \text{and} \qquad 1-\left(k_1+k_3+\frac{2c_0k_1k_3}{k_2-g_0}\right)N_2>0.$$

Next, we select l small enough so that

$$(k_1+k_3)N_2-l\left(\frac{1}{2}+\frac{k_3}{k_1}+\frac{|k_3-k_1|}{2}N_2\right)>0,\ (k_2-g_0)k_3N_2-\frac{l}{k_1}\left(\frac{k_2^2}{2}+g_0^2\right)>0,$$

and

$$1-\left(k_1+k_3+\frac{2c_0k_1k_3}{k_2-g_0}\right)N_2-l>0.$$

After that, we pick N_1 very large so that

$$N_1 \int_0^{t_0} g(s)ds - 3k_3 N_2 - \frac{c(1+N_2)^2}{l\rho_1\rho_2} > 0.$$

Therefore, we have

$$\begin{aligned}\mathcal{L}'(t) \le & -(\beta_0 - c\delta)E(t) + \left(N - \frac{c}{\delta}\right)E'(t) \\ & + \frac{c}{\delta}\Big[(1+N_1) + C_\alpha(1+N_1+5k_3N_2)\Big](h\circ\psi_x)(t) \\ & + \left(\frac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \varphi_t\psi_{xt}dx + \rho_1\left(\frac{k_3}{k_1} - 1\right)\int_0^L \varphi_{xt}w_t dx,\end{aligned}$$

for some $\beta_0 > 0$. At this point, we take $\delta < \frac{\beta_0}{c}$. Consequently, we obtain, for some $m > 0$,

$$\begin{aligned}\mathcal{L}'(t) \le & -mE(t) + (N-c)E'(t) + c\Big[(1+N_1) + C_\alpha(1+N_1+5k_3N_2)\Big](h\circ\psi_x)(t) \\ & + \left(\frac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \varphi_t\psi_{xt}dx + \rho_1\left(\frac{k_3}{k_1} - 1\right)\int_0^L \varphi_{xt}w_t dx, \qquad \forall\, t \ge t_0.\end{aligned}$$

Finally, we choose N so large that $\mathcal{L} \sim E$ and $\frac{N}{2} > c$, therefore we have, $\forall\, t \ge t_0$,

$$\begin{aligned}\mathcal{L}'(t) \le & -mE(t) + \frac{\alpha}{4}N(g\circ\psi_x)(t) \\ & - \left[\frac{N}{4} - c(1+N_1) - cC_\alpha(1+N_1+5k_3N_2)\right](h\circ\psi_x)(t) \\ & + \left(\frac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \varphi_t\psi_{xt}dx + \rho_1\left(\frac{k_3}{k_1} - 1\right)\int_0^L \varphi_{xt}w_t dx.\end{aligned}$$

As $\frac{\alpha g^2}{\alpha g - g'} < g$, it follows from (**A.1**) and the Lebesgue Dominated Convergence Theorem that

$$\lim_{\alpha\to 0^+} \alpha C_\alpha = \lim_{\alpha\to 0^+} \int_0^\infty \frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} ds = 0.$$

Consequently, there exists $\alpha_0 \in (0,1)$ such that

$$\alpha C_\alpha < \frac{m}{8c\,(1 + N_1 + 5k_3 N_2)} \left(2 + \frac{k_2 - g_0}{6g_0}\right), \qquad \text{whenever} \qquad \alpha < \alpha_0.$$

Then, choose N even larger (if needed) so that

$$N > \max\left\{8c(1+N_1),\ \frac{m}{\alpha_0}\left(2 + \frac{k_2 - g_0}{6g_0}\right)\right\}$$

and set

$$\alpha = \frac{m}{N}\left(2 + \frac{k_2 - g_0}{6g_0}\right).$$

So

$$\frac{1}{8}N - c(1+N_1) > 0 \qquad \text{and} \qquad \alpha = \frac{m}{N}\left(2 + \frac{k_2 - g_0}{6g_0}\right) < \alpha_0.$$

This yields

$$\begin{aligned}\frac{1}{4}N - c(1+N_1) - cC_\alpha\,(1 + N_1 + 5k_3N_2) >& \frac{1}{4}N - c(1+N_1) - \frac{m}{8\alpha}\left(2 + \frac{k_2 - g_0}{6g_0}\right)\\ =& \frac{1}{8}N - c(1+N_1) > 0.\end{aligned}$$

Hence, we arrive at estimate (3.11). ∎

3.3 General Decay Rates for Equal Speeds of Wave Propagation

In this section, we state and prove a general decay result under equal speeds of wave propagation condition. The exponential and polynomial decay results are only special cases.

Theorem 3.2 *Let* $(\varphi_0, \varphi_1) \in H_0^1(0,L) \times L^2(0,L)$ *and* (ψ_0, ψ_1), $(w_0, w_1) \in H_*^1(0,L) \times L_*^2(0,L)$. *Assume that* (**A.1**) *and* (**A.2**) *hold and that*

$$\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} \qquad \text{and} \qquad k_1 = k_3. \tag{3.12}$$

Then for l *small enough, the solution of* (P_2) *satisfies, for* $t > t_0$,

$$E(t) \leq C \exp\left(-\lambda \int_0^t \xi(s)ds\right), \qquad \text{for} \;\; p = 1, \tag{3.13}$$

and

$$E(t) \leq C\left(1 + \int_0^t \xi(s)ds\right)^{-\frac{1}{p-1}}, \qquad \text{for} \;\; 1 < p < 2, \tag{3.14}$$

where $C > 0$ *is a constant independent of* t *but may depend on the initial data and* $\lambda > 0$ *is a constant independent of both* t *and the initial data.*

Proof. Using (3.12), estimate (3.11) becomes

$$\mathcal{L}'(t) \leq -mE(t) + \frac{m}{2}\left(1 + \frac{k_2 - g_0)}{12 g_0}\right)(g \circ \psi_x)(t), \qquad \forall\, t \geq t_0. \tag{3.15}$$

Case $p = 1$. Multiply both sides of estimate (3.15) by ξ, then use assumption (**A.2**) and Lemma 3.2 to get

$$\xi(t)\mathcal{L}'(t) \leq -m\xi(t)E(t) - cE'(t), \qquad \forall\, t \geq t_0.$$

The non-increasing property of ξ gives

$$(\xi\mathcal{L} + cE)'(t) \leq -m\xi(t)E(t), \qquad \forall\, t \geq t_0.$$

A simple integration over (t_0, t) together with the fact that $\xi\mathcal{L} + cE \sim E$ gives

$$E(t) \leq C \exp\left(-\lambda \int_{t_0}^{t} \xi(s)ds\right), \qquad \forall\, t > t_0.$$

Case $\boldsymbol{1 < p < 2}$. From Lemma 3.9–3.10 we deduce that, the functional $\mathscr{L}$ defined by

$$\mathscr{L}(t) := \mathcal{L}(t) + \frac{m(k_2 - g_0)}{8g_0} J(t)$$

is nonnegative and satisfies

$$\mathscr{L}'(t) \leq -\beta E(t), \qquad \forall\, t \geq t_0,$$

where $\beta > 0$. Thus,

$$\int_0^\infty E(t)dt < \infty. \tag{3.16}$$

Define a functional η by

$$\eta(t) := \int_0^t \|\psi_x(t) - \psi_x(t-s)\|_2^2 ds,$$

it follows from (3.16) that

$$\begin{aligned}
\eta(t) &= \int_0^t \|\psi_x(t) - \psi_x(s)\|_2^2 ds \le 2\int_0^t \big(\|\psi_x(t)\|_2^2 + \|\psi_x(s)\|_2^2\big) ds \\
&\le \frac{4}{l}\int_0^t \big(E(t) - E(s)\big) ds \le \frac{8}{l}\int_0^t E(s) ds < \infty \qquad \forall\, t \ge 0.
\end{aligned}$$

Also, there exists $t_1 > 0$ such that $\eta(t_1) > 0$, otherwise $\eta(t) = 0$ for any $t \ge 0$. This gives

$$(g \circ \psi_x)(t) \le g(0)\int_0^t \|\psi_x(x) - \psi_x(t-s)\|_2^2 ds = 0, \qquad \forall\, t \ge 0.$$

Hence, we get an exponential decay from (3.15). Without loss of generality, we assume $t_1 = t_0$, using Jensen's inequality, condition (**A.2**) and Lemma 3.2 estimate (3.15) becomes

$$\begin{aligned}
\mathcal{L} &\le -mE(t) + c\eta(t) \cdot \frac{1}{\eta(t)} \int_0^t g^{p\cdot\frac{1}{p}}(t-s)\|\psi_x(t) - \psi_x(s)\|_2^2 ds \\
&\le -mE(t) + c\eta(t)\left(\frac{1}{\eta(t)}\int_0^t g^p(t-s)\|\psi_x(t) - \psi_x(s)\|_2^2 ds\right)^{\frac{1}{p}} \\
&\le -mE(t) + c\eta^{1-1/p}(t)\left(-\int_0^t \frac{g'(t-s)}{\xi(t-s)}\|\psi_x(t) - \psi_x(s)\|_2^2 ds\right)^{\frac{1}{p}} \\
&\le -mE(t) + c\left(-\frac{1}{\xi(t)}\int_0^t g'(t-s)\|\psi_x(t) - \psi_x(s)\|_2^2 ds\right)^{\frac{1}{p}} \\
&\le -mE(t) + c\left(-\frac{E'(t)}{\xi(t)}\right)^{\frac{1}{p}}, \qquad \forall\, t \ge t_0.
\end{aligned}$$

This last estimate together with Lemma 3.2 and Young's inequality yield, for any $\varepsilon > 0$,

$$\begin{aligned}
\big(E^{p-1}\mathcal{L}\big)'(t) &= (p-1)E^{p-2}(t)E'(t)\mathcal{L}(t) + E^{p-1}(t)\mathcal{L}'(t) \\
&\leq -mE^p(t) + cE^{p-1}(t)\left(-\frac{E'(t)}{\xi(t)}\right)^{\frac{1}{p}} \\
&\leq -(m-\varepsilon)E^p(t) - \frac{c}{\varepsilon}\frac{E'(t)}{\xi(t)}, \qquad \forall\, t \geq t_0.
\end{aligned}$$

By taking $\varepsilon < m$, we obtain, for some fixed $m_0 > 0$,

$$\xi(t)\big(E^{p-1}\mathcal{L}\big)'(t) \leq -m_0\xi(t)E(t) - cE'(t), \qquad \forall\, t \geq t_0.$$

Set $\mathcal{F} = \xi E^{p-1}\mathcal{L} + cE \sim E$, then the nonincreasing property of ξ gives, for any $t \geq t_0$,

$$\mathcal{F}'(t) \leq -m_0\xi(t)E^p(t).$$

We integrate over (t_0, t) to get

$$E(t) \leq C\left(1 + \int_{t_0}^{t}\xi(s)ds\right)^{-\frac{1}{p-1}}, \qquad \forall\, t > t_0,$$

where C is a positive constant. ∎

Remark 3.1 *The smallness condition on l makes the Bresse system close to Timoshenko system and, hence, inherits some of its stability properties.*

Example 3.1 *Let $g(t) = \dfrac{a}{(1+t)^q}$ with $q > 1$, and $a > 0$ is to be chosen so that (**A.1**)* *is satisfied. Then*

$$g'(t) = -a_0 \left(\frac{a}{(1+t)^q} \right)^{\frac{q+1}{q}} = -\xi(t) g^p(t),$$

with $\xi(t) = a_0 = \dfrac{q}{a^{1/q}}$ *and* $p = \dfrac{q+1}{q} < 2$. *Therefore, for a fixed* $t_0 > 0$, *inequality* (3.14) *entails that there exists* $C > 0$ *such that, for* t *large,*

$$E(t) \leq C \left(1 + \int_{t_0}^{t} \xi(s) ds \right)^{-\frac{1}{p-1}} = \frac{c}{(1+t)^q},$$

with the optimal decay rate q. *For more examples, see [22].*

3.4 General Decay Rate for Different Speeds of Wave Propagation

In this section, we state and prove a generalized decay result in the case of non-equal speeds of wave propagation. We start by differentiating both sides of the differential equations in (P_2) with respect to t and use the fact that

$$\begin{aligned} \frac{\partial}{\partial t} \left[\int_0^t g(t-s) \psi_{xx}(s) ds \right] &= \frac{\partial}{\partial t} \left[\int_0^t g(s) \psi_{xx}(t-s) ds \right] \\ &= \int_0^t g(t-s) \psi_{xxt}(s) ds + g(t) \psi_{0xx}, \end{aligned}$$

to obtain the following system

$$\begin{cases} \rho_1 \varphi_{ttt} - k_1(\varphi_{xt} + \psi_t + l w_t)_x - l k_3 (w_{xt} - l \varphi_t) = 0, \\ \rho_2 \psi_{ttt} - k_2 \psi_{xxt} + k_1(\varphi_{xt} + \psi_t + l w_t) + \displaystyle\int_0^t g(t-s) \psi_{xxt}(s) ds + g(t) \psi_{0xx} = 0, \\ \rho_1 w_{ttt} - k_3 (w_{xt} - l \varphi_t)_x + l k_1 (\varphi_{xt} + \psi_t + l w_t) = 0. \end{cases} \tag{P_2'}$$

The "second" energy functional associated to (P_2) (which is the energy functional associated to (P_2')) is given by

$$E_*(t) := \frac{1}{2}\int_0^L \left[\rho_1\varphi_{tt}^2 + \rho_2\psi_{tt}^2 + \rho_1 w_{tt}^2 + \left(k_2 - \int_0^t g(s)ds\right)\psi_{xt}^2 \right.$$
$$\left. + k_3(w_{xt} - l\varphi_t)^2 + k_1(\varphi_{xt} + \psi_t + lw_t)^2\right] dx + \frac{1}{2}(g \circ \psi_{xt})(t), \quad \forall\, t \geq 0. \tag{3.17}$$

We have the following result due to [77, Lemma 3.11].

Lemma 3.11 *Let (φ, ψ, w) be the strong solution of (P_2). Then, the second energy of (P_2) satisfies, for all $t \geq 0$,*

$$E_*'(t) = -\frac{1}{2}g(t)\int_0^L \psi_{xt}^2 dx + \frac{1}{2}(g' \circ \psi_{xt}) - g(t)\int_0^L \psi_{tt}\psi_{0xx} dx \tag{3.18}$$

and

$$E_*(t) \leq c\left(E_*(0) + \int_0^L \psi_{0xx}^2 dx\right). \tag{3.19}$$

Corollary 3.1 *Let (φ, ψ, w) be the strong solution of (P_2). Then,*

$$0 \leq -(g' \circ \psi_{xt})(t) \leq c\big(-E_*'(t) + c_1 g(t)\big), \qquad \forall\, t \geq 0, \tag{3.20}$$

where c_1 is some fixed positive constant.

Proof. From equation (3.18) and inequality (3.19) we have

$$0 \leq -g' \circ \psi_{xt} \;=\; -2E_*'(t) - g(t)\int_0^L \psi_{xt}^2 dx - 2g(t)\int_0^L \psi_{tt}\psi_{0xx} dx$$

$$\begin{aligned}
&\leq -2E_*'(t) - 2g(t)\int_0^L \psi_{tt}\psi_{0xx}dx \\
&\leq -2E_*'(t) + g(t)\int_0^L (\psi_{tt}^2 + \psi_{0xx}^2)dx \\
&\leq -2E_*'(t) + g(t)\left(\frac{2}{\rho_2}E_*(t) + \int_0^L \psi_{0xx}^2 dx\right) \\
&\leq c\left(-E_*'(t) + c_1 g(t)\right),
\end{aligned}$$

for some positive constant c_1. ∎

Now we estimate the third term in the right-hand side of (3.11) as in [77].

Lemma 3.12 *Let (φ, ψ, w) be the strong solution of (P_2). Then, for any $\varepsilon > 0$, we have*

$$\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right)\int_0^L \varphi_t \psi_{xt} dx \leq \varepsilon E(t) + \frac{c}{\varepsilon}(g \circ \psi_{xt} - E'(t) + g(t)), \quad \forall t \geq t_0. \tag{3.21}$$

Proof.

$$\begin{aligned}
\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right)\int_0^L \varphi_t \psi_{xt} dx \;=\; & \frac{\left(\frac{\rho_1 k_2}{k_2} - \rho_2\right)}{\int_0^t g(s)ds}\int_0^L \varphi_t \int_0^t g(t-s)(\psi_{xt}(t) - \psi_{xt}(s))dsdx \\
& + \frac{\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right)}{\int_0^t g(s)ds}\int_0^L \varphi_t \int_0^t g(t-s)\psi_{xt}(s)dsdx.
\end{aligned} \tag{3.22}$$

By observing that $\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds$, for all $t \geq t_0$ and exploiting Young's inequality and Lemma 2.3 (for ψ_{xt}), we get, for $\varepsilon > 0$ and $t \geq t_0$,

$$\frac{\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right)}{\int_0^t g(s)ds}\int_0^L \varphi_t \int_0^t g(t-s)(\psi_{xt}(t) - \psi_{xt}(s))dsdx \leq \frac{\varepsilon}{4}\rho_1 \int_0^L \varphi_t^2 dx + \frac{c}{\varepsilon}(g \circ \psi_{xt}).$$

On the other hand, using integration by parts, Young and Hölder's inequalities together with the non-increasing property of E, we get, for any $\varepsilon > 0$,

$$
\begin{aligned}
&\frac{\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right)}{\int_0^t g(s)ds} \int_0^L \varphi_t \int_0^t g(t-s)\psi_{xt}(s)dsdx \\
&= \frac{\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right)}{\int_0^t g(s)ds} \int_0^L \varphi_t \left(g(0)\psi_x - g(t)\psi_{0x} + \int_0^t g'(t-s)\psi_x(s)ds \right) dx \\
&= \frac{\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right)}{\int_0^t g(s)ds} \int_0^L \varphi_t \left(g(t)(\psi_x - \psi_{0x}) - \int_0^t g'(t-s)(\psi_x(t) - \psi_x(s))ds \right) dx \\
&\leq \frac{\varepsilon}{4}\rho_1 \int_0^L \varphi_t^2 dx + \frac{c}{\varepsilon} g(t) \int_0^L (\psi_x^2 + \psi_{0x}^2)dx - \frac{c}{\varepsilon} g' \circ \psi_x \\
&\leq \frac{\varepsilon}{4}\rho_1 \int_0^L \varphi_t^2 dx + \frac{c}{\varepsilon} E(0) g(t) - \frac{c}{\varepsilon} g' \circ \psi_x.
\end{aligned}
$$

Inserting the last two inequalities in (3.22), we get (3.21). ∎

Theorem 3.3 *Let*

$$
(\varphi_0, \varphi_1) \in \left(H^2(0,L) \cap H_0^1(0,L)\right) \times H_0^1(0,L)
$$

and

$$
(\psi_0, \psi_1),\ (w_0, w_1) \in (H_*^2(0,L) \cap H_*^1(0,L)) \times H_*^1(0,L).
$$

*Assume that conditions (**A.1**), (**A.2**) hold and that*

$$
\frac{\rho_1}{k_1} \neq \frac{\rho_2}{k_2} \qquad \text{and} \qquad k_1 = k_3.
$$

Then for l small enough and for any $t_0 > 0$, there exists a positive constant C that may depend on the initial data but independent of t, for which the strong solution of

(P_2) *satisfies*

$$E(t) \leq Ct^{1-1/p} \left(\int_{t_0}^{t} \xi(s) ds \right)^{-\frac{1}{p}}, \qquad \forall\, t > t_0. \tag{3.23}$$

Proof. Exploiting Lemma (3.12) in estimate (3.11), we have, for some $m > 0$,

$$\begin{aligned}
\mathcal{L}'(t) \leq & - mE(t) + c(g \circ \psi_x)(t) + \left(\frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_0^L \varphi_t \psi_{xt} dx \\
\leq & - (m - \varepsilon) E(t) + c(g \circ \psi_x)(t) + \frac{c}{\varepsilon} \big(g \circ \psi_{xt}(t) - E'(t) + g(t) \big), \quad \forall\, t \geq t_0.
\end{aligned}$$

After fixing ε small enough, we arrive at

$$\mathcal{L}'(t) \leq -m_1 E(t) + c\big(g \circ \psi_x + g \circ \psi_{xt} \big)(t) - cE'(t) + cg(t), \quad \forall\, t \geq t_0,$$

where m_1 is a fixed positive constant. By setting $\mathcal{F} := \mathcal{L} + cE \sim E$, we obatain, for any $t \geq t_0$,

$$\mathcal{F}'(t) \leq -m_1 E(t) + c\big(g \circ \psi_x + g \circ \psi_{xt} \big)(t) + cg(t). \tag{3.24}$$

Case $\boldsymbol{p = 1}$. Multiplying both sides of estimate (3.24) by $\xi(t)$, then using hypothesis (**A.2**) and Corollary 3.1 we get, for any $t \geq t_0$,

$$\begin{aligned}
\xi(t)\mathcal{F}'(t) &\leq -m_1 \xi(t) E(t) + c\xi(t) \big(g \circ \psi_x + g \circ \psi_{xt} \big)(t) + c\xi(t) g(t) \\
&\leq -m_1 \xi(t) E(t) + c\xi(t) \left(-\frac{g'}{\xi} \circ \psi_x - \frac{g'}{\xi} \circ \psi_{xt} \right)(t) + c\xi(t) g(t) \\
&\leq -m_1 \xi(t) E(t) - cE'(t) + c\big(-E'_*(t) + c_1 g(t) \big) + c\xi(0) g(t).
\end{aligned}$$

From the non-increasing property of ξ, we have, for some fixed positive constant c_2,

$$\big(\xi\mathcal{F} + cE + cE_*\big)'(t) \leq -m_1\xi(t)E(t) + c_2 g(t), \quad \forall\, t \geq t_0,$$

which implies

$$m_1\xi(t)E(t) \leq -\big(\xi\mathcal{F} + cE + cE_*\big)'(t) + c_2 g(t), \quad \forall\, t \geq t_0.$$

An integration over (t_0, t), exploitation of the non-increasing property of E and estimate (3.19) yield, for any $t > t_0$,

$$\begin{aligned} m_1 E(t)\int_{t_0}^{t}\xi(s)ds &\leq -\,\big(\xi\mathcal{F} + cE + cE_*\big)(t) + \big(\xi\mathcal{F} + cE + cE_*\big)(t_0) + c_2\int_{t_0}^{t} g(s)ds \\ &\leq \big(\xi\mathcal{F} + cE + cE_*\big)(0) + c\int_0^L \psi_{0xx}^2 dx + c_2(b - l). \end{aligned}$$

Thus, we have, for some fixed positive constant C,

$$E(t) \leq C\left(\int_{t_0}^{t}\xi(s)ds\right)^{-1}, \qquad \forall\, t > t_0.$$

Case $\boldsymbol{1 < p < 2}$. First, we introduce a functional η defined by

$$\eta(t) := \frac{1}{t}\int_0^t \big(\|\psi_x(t) - \psi_x(s)\|_2^2 + \|\psi_{xt}(t) - \psi_{xt}(s)\|_2^2\big)ds, \quad \forall\, t > 0.$$

Then, it follows from (3.1), (3.2), (3.17) and (3.19) that

$$\eta(t) \;=\; \frac{1}{t}\int_0^t \big(\|\psi_x(t) - \psi_x(s)\|_2^2 + \|\psi_{xt}(t) - \psi_{xt}(s)\|_2^2\big)ds$$

$$
\begin{aligned}
&\le \frac{4}{lt}\int_0^t \big(E(t)+E(t-s)+E_*(t)+E_*(t-s)\big)ds \\
&\le \frac{8}{lt}\int_0^t \left[E(0)+c\left(E_*(0)+\int_0^L \psi_{0xx}^2 dx\right)\right]ds \\
&\le \frac{8}{l}\left[E(0)+c\left(E_*(0)+\int_0^L \psi_{0xx}^2 dx\right)\right] < +\infty, \quad \forall\, t>0. \qquad (3.25)
\end{aligned}
$$

There exists $t_1 > 0$ such that $\eta(t) > 0$ for any $t \ge t_1$, otherwise $\eta(t) = 0$ for any $t \ge 0$, which implies $(g \circ \psi_x)(t) = (g \circ \psi_{xt})(t) = 0$ for any $t \ge 0$. Hence, we obtain the following decay rate from (3.24)

$$
E(t) \le C\left(\int_{t_0}^t \xi(s)ds\right)^{-1}, \qquad \forall\, t > t_0.
$$

Without loss of generality, we assume $t_1 = t_0$, then it follows from (**A.2**), (3.2), (3.20), (3.25) and Jensen's inequality that

$$
\begin{aligned}
&(g\circ\psi_x + g\circ\psi_{xt})(t) \\
&\quad = t\eta(t)\cdot\frac{1}{\eta(t)}\int_0^t g^{p\cdot\frac{1}{p}}(t-s)\cdot\frac{1}{t}\big[\|\psi_x(t)-\psi_x(s)\|_2^2+\|\psi_{xt}(t)-\psi_{xt}(s)\|_2^2\big]ds \\
&\quad \le t\eta(t)\left(\frac{1}{t\eta(t)}\int_0^t g^p(t-s)\big[\|\psi_x(t)-\psi_x(s)\|_2^2+\|\psi_{xt}(t)-\psi_{xt}(s)\|_2^2\big]ds\right)^{1/p} \\
&\quad \le t\eta^{1-1/p}(t)\left(-\frac{1}{t}\int_0^t \frac{g'(t-s)}{\xi(t-s)}\big[\|\psi_x(t)-\psi_x(s)\|_2^2+\|\psi_{xt}(t)-\psi_{xt}(s)\|_2^2\big]ds\right)^{1/p} \\
&\quad \le ct\left(-\frac{1}{t\xi(t)}\int_0^t g'(t-s)\big[\|\psi_x(t)-\psi_x(s)\|_2^2+\|\psi_{xt}(t)-\psi_{xt}(s)\|_2^2\big]ds\right)^{1/p} \\
&\quad \le ct\left(\frac{1}{t\xi(t)}\big[-E'(t)+c\big(-E_*'(t)+c_1 g(t)\big)\big]\right)^{1/p}, \qquad \forall\, t\ge t_0.
\end{aligned}
$$

Therefore, estimates (3.24) becomes, for any $t \geq t_0$,

$$\mathcal{F}'(t) \leq -m_1 E(t) + ct \left(\frac{1}{t\xi(t)} \big[-E'(t) + c\big(-E'_*(t) + c_1 g(t) \big) \big] \right)^{1/p} + cg(t). \quad (3.26)$$

Next, we define a functional $\mathcal{F}_1$ by

$$\mathcal{F}_1(t) := \left(\frac{E(t)}{t} \right)^{p-1} \mathcal{F}(t), \qquad \forall\, t \geq t_0.$$

Then, estimate (3.26) together with the (3.2) and Young's inequality yield, for any $\varepsilon > 0$ and for all $t \geq t_0$,

$$\begin{aligned}
\mathcal{F}_1'(t) &= (p-1) \left(\frac{E(t)}{t} \right)^{p-2} \left(-\frac{E(t)}{t^2} + \frac{E'(t)}{t} \right) \mathcal{F}_1(t) + \left(\frac{E(t)}{t} \right)^{p-1} \mathcal{F}_1'(t) \\
&\leq -m_1 E(t) \left(\frac{E(t)}{t} \right)^{p-1} + c \left(\frac{E(t)}{t} \right)^{p-1} g(t) \\
&\quad + ct \left(\frac{E(t)}{t} \right)^{p-1} \left(\frac{1}{t\xi(t)} \big[-E'(t) + c\big(-E'_*(t) + c_1 g(t) \big) \big] \right)^{1/p} \\
&\leq -m_1 t \left(\frac{E(t)}{t} \right)^{p} + c \left(\frac{E(0)}{t_0} \right)^{p-1} g(t) \\
&\quad + ct \left(\frac{E(t)}{t} \right)^{p-1} \left(\frac{1}{t\xi(t)} \big[-E'(t) + c\big(-E'_*(t) + c_1 g(t) \big) \big] \right)^{1/p} \\
&\leq -(m_1 - \varepsilon) t \left(\frac{E(t)}{t} \right)^{p} + \frac{c}{\varepsilon \xi(t)} \big(-E'(t) - cE'_*(t) + c_2 g(t) \big) + cg(t),
\end{aligned}$$

where $c_2 > 0$ is a constant. We pick ε so that $m_1 - \varepsilon > 0$ and multiply the above inequality by $\xi(t)$, we get, for some c_3, $m_2 > 0$,

$$\xi(t)\mathcal{F}_1'(t) \leq -m_2 t\xi(t) \left(\frac{E(t)}{t} \right)^{p} - cE'(t) - cE'_*(t) + c_3 g(t), \qquad \forall\, t \geq t_0.$$

Let $\mathcal{F}_2 = \xi\mathcal{F}_1 + cE + cE_* \geq cE + cE'_*$, the nonincreasing property of ξ implies

$$\mathcal{F}'_2(t) \leq -m_2 t\xi(t)\left(\frac{E(t)}{t}\right)^p + cg(t), \qquad \forall\, t \geq t_0.$$

Since the map $t \mapsto t\left(\frac{E(t)}{t}\right)^p$ is nonincreasing, we have, for ant $t \geq t_0$,

$$\begin{aligned} m_2 t\left(\frac{E(t)}{t}\right)^p \int_{t_0}^{t} \xi(s)ds &\leq m_2 \int_{t_0}^{t} s\left(\frac{E(s)}{s}\right)^p \xi(s)ds \leq -\int_{t_0}^{t} \big(\mathcal{F}_2(s) + c_3 g(s)\big)ds \\ &\leq -\mathcal{F}_2(t) + \mathcal{F}_2(t_0) + c_3(b-l) \leq \mathcal{F}_2(t_0) + c_3(b-l). \end{aligned}$$

This last estimate implies that

$$E(t) \leq Ct^{1-1/p}\left(\int_{t_0}^{t} \xi(s)ds\right)^{-1/p}, \qquad \forall\, t > t_0.$$

This completes the proof of Theorem 3.3. ▮

Example 3.2 *(1) Consider the relaxation function $g(t) = a\exp(-\alpha t)$, where a, α are positive constants and a is chosen so that hypothesis (**A.1**) is satisfied, then*

$$g'(t) = -\alpha H(g(t)) \qquad \text{with} \qquad H(s) = s.$$

Therefore, $H_2(t) = t$ and estimate (3.23) *implies that there exist $t_1 > 0$ such that*

$$E(t) \leq \frac{c}{t - t_1}, \qquad \forall\, t > t_1.$$

*(2) Consider $g(t) = ae^{-(1+t)^\nu}$, for $0 < \nu < 1$ and a is chosen so that condition (**A.1**)*

is satisfied, then

$$g'(t) = -\xi(t)H(g(t)) \qquad \text{with} \qquad \xi(t) = \nu(1+t)^{\nu-1} \qquad \text{and} \qquad H(s) = s.$$

Therefore $H_2(t) = t$ and estimate (3.23) entails that

$$E(t) \leq \frac{c}{(1+t)^{\nu}}, \qquad \text{for} \quad t \quad \text{large} \quad \text{enough.}$$

(3) *Consider the following relaxation function, for $\nu > 1$,*

$$g(t) = \frac{a}{(1+t)^{\nu}}$$

*and a is chosen so that hypothesis (**A.1**) remains valid. Then*

$$g'(t) = -bH(g(t)) \qquad \text{with} \qquad H(s) = s^{p},$$

where b is a fixed constant, $p = \frac{1+\nu}{\nu}$ and it satisfies $1 < p < 2$. Then, $H_2(t) = pt^{p}$ and we deduce from (3.23) that there exist $t_1 > 0$ such that

$$E(t) \leq \frac{c}{(1+t)^{(2-p)/p}} = \frac{c}{(1+t)^{(\nu-1)/(\nu+1)}}, \qquad \forall\, t > t_1.$$

CHAPTER 4

ON THE GENERAL DECAY FOR A SYSTEM OF VISCOELASTIC WAVE EQUATIONS

In this Chapter, we consider the following coupled system of viscoelastic wave equations:

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g_1(t-s)\Delta u(\cdot,s)ds + f_1(u,v) = 0, & \text{in } \Omega \times (0,+\infty), \\ v_{tt} - \Delta v + \displaystyle\int_0^t g_2(t-s)\Delta v(\cdot,s)ds + f_2(u,v) = 0, & \text{in } \Omega \times (0,+\infty), \\ u = v = 0, & \text{on } \partial\Omega \times (0,+\infty), \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, \quad v(\cdot,0) = v_0, \quad v_t(\cdot,0) = v_1, & \text{in } \Omega, \end{cases} \tag{P_3}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $u_0,\ u_1,\ v_0,\ v_1$ are given initial data, $g_1,\ g_2$ are the relaxation functions and $f_1,\ f_2$ are nonlinear functions. This Chapter is organized as follows: in Section 4.1, we state some preliminary results and our main result. In Section 4.2, we state and prove some technical lemmas needed for the entire work. We give the proof of our main result and some comments in Section 4.3.

4.1 Preliminaries

In this section, we state the existence theorem and present some useful lemmas. We stark with following remarks concerning assumptions (**A.3**)–(**A.5**).

Remark 4.1

*(1) It follows from assumption (**A.3**) that, for $i = 1, 2$,*

$$\lim_{t\to+\infty} g_i(t) = 0 \qquad \text{and} \qquad g_i(t) \le \frac{1-l_i}{t}, \qquad \forall\, t > 0.$$

*Also, the assumption (**A.4**) entails that, there exists $t_i > 0$ (for $i = 1, 2$) such that*

$$g_i(t_i) = r \qquad \text{and} \qquad g_i(t) \le r, \qquad \forall\, t \ge t_0 := \max\{t_1, t_2\}.$$

The non-increasing property of g_i gives

$$0 < g_i(t_i) \le g_i(t) \le g_i(0), \qquad \forall\, t \in [0, t_0].$$

A combination of this with the continuity of H_i yields (for $i = 1, 2$)

$$a_i \le H_i(g_i(t)) \le b_i, \qquad \forall\, t \in [0, t_0],$$

for some constants $a_i, b_i > 0$, $i = 1, 2$. *Consequently, for any* $t \in [0, t_0]$ *and for* $i = 1, 2$, *we have*

$$g_i'(t) \leq -\xi_i(t) H_i(g_i(t)) \leq -a_i \xi_i(t) = -\frac{a_i}{g_i(0)} \xi_i(t) g_i(0) \leq -\frac{a_i}{g_i(0)} \xi_i(t) g_i(t).$$

This implies that, for $i = 1, 2$,

$$\xi_i(t) g_i(t) \leq -\frac{g_i(0)}{a_i} g_i'(t), \qquad \forall\, t \in [0, t_0]. \tag{4.1}$$

(2) If H *is a strictly increasing and strictly convex* C^2-*function on* $(0, r]$, *with* $H(0) = H'(0) = 0$, *then it has an extension* $\bar{H}$ *which is a strictly increasing and strictly convex* C^2-*function on* $(0, +\infty)$. *For instance, we can define* $\bar{H}$, *for any* $t > r$, *by*

$$\bar{H}(t) := \frac{H''(r)}{2} t^2 + \big(H'(r) - H''(r) r\big) t + \left(H(r) + \frac{H''(r)}{2} r^2 - H'(r) r\right).$$

(3) Inequality (1.34) *yields, for some positive constant* k, *that*

$$|f_i(x, y)| \leq k \left(|x| + |y| + |x|^{\beta_i} + |y|^{\beta_i}\right) \tag{4.2}$$

for all $(x, y) \in \mathbb{R}^2$ *and* $i = 1, 2$.

For completeness, we state, without proof, the global existence and regularity result whose proof can be found in [70].

Theorem 4.1 *Let* $(u_0, u_1), (v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *be given. Assume that hypotheses (***A.3***) and (***A.5***) are satisfied. Then, problem* (P_3) *has a unique weak solution*

$$u,\, v \in C\big([0,\infty); H_0^1(\Omega)\big) \cap C^1\big([0,\infty); L^2(\Omega)\big) \cap C^2\big([0,\infty); H^{-1}(\Omega)\big).$$

Moreover, if $(u_0, u_1), (v_0, v_1) \in \big(H^2(\Omega) \cap H_0^1(\Omega)\big) \times H_0^1(\Omega)$*, then problem* (P_3) *has a unique strong solution*

$$u,\, v \in L^\infty\big([0,\infty); H^2(\Omega) \cap H_0^1(\Omega)\big) \cap W^{1,\infty}\big([0,\infty); H_0^1(\Omega)\big) \cap W^{2,\infty}\big([0,\infty); L^2(\Omega)\big).$$

Now, we introduce the energy functional

$$\begin{aligned} E(t) := \frac{1}{2}\left[\|u_t\|_2^2 + \left(1 - \int_0^t g_1(s)ds\right)\|\nabla u\|_2^2 + (g_1 \circ \nabla u)(t)\right] \\ + \frac{1}{2}\left[\|v_t\|_2^2 + \left(1 - \int_0^t g_2(s)ds\right)\|\nabla v\|_2^2 + (g_2 \circ \nabla v)(t)\right] \\ + \int_\Omega F(u,v)dx, \end{aligned} \tag{4.3}$$

where, for any $w \in L^2_{loc}\big([0,+\infty); L^2(\Omega)\big)$ and $i = 1, 2$,

$$(g_i \circ w)(t) := \int_0^t g_i(t-s)\|w(t) - w(s)\|_2^2 ds.$$

Lemma 4.1 *Let (u, v) be the solution of (P_3). Then,*

$$E'(t) = -\frac{1}{2}g_1(t)\|\nabla u\|_2^2 + \frac{1}{2}(g_1' \circ \nabla u)(t) - \frac{1}{2}g_2(t)\|\nabla v\|_2^2 + \frac{1}{2}(g_2' \circ \nabla v)(t) \leq 0, \ \forall\, t \geq 0. \tag{4.4}$$

As in [23], we set, for any $0 < \alpha < 1$ and for $i = 1, 2$,

$$C_{\alpha,i} := \int_0^\infty \frac{g_i^2(s)}{\alpha g_i(s) - g_i'(s)}ds \qquad \text{and} \qquad h_i(t) := \alpha g_i(t) - g_i'(t).$$

Lemma 4.2 ([23]) - *Assume that conditions (**A.3**) holds. Then for any $w \in L^2_{loc}\big([0, +\infty); L^2(\Omega)\big)$, we have, for $i = 1, 2$,*

$$\int_\Omega \left(\int_0^t g_i(t-s)(w(t) - w(s))ds\right)^2 dx \leq C_{\alpha,i}(h_i \circ w)(t), \qquad \forall\, t \geq 0. \tag{4.5}$$

We will also need the following embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$, for $q \geq 2$ if $n = 1, 2$ or $2 \leq q \leq \dfrac{2n}{n-2}$ if $n \geq 3$, that is,

$$\|w\|_q \leq c\|\nabla w\|_2, \qquad \forall\, w \in H_0^1(\Omega). \tag{4.6}$$

4.2 Technical Lemmas

In this section, we state and prove some lemmas needed to establish our main result.

Lemma 4.3 *Assume that* (**A.3**) – (**A.5**) *hold. Then, the functional I defined by*

$$I(t) := \int_\Omega u u_t dx + \int_\Omega v v_t dx$$

satisfies, along the solution of (P_3), *the estimates*

$$\begin{aligned} I'(t) &\leq \|u_t\|_2^2 - \frac{l_1}{2}\|\nabla u\|_2^2 + cC_{\alpha,1}(h_1 \circ \nabla u)(t) + \|v_t\|_2^2 \\ &\quad -\frac{l_2}{2}\|\nabla v\|_2^2 + cC_{\alpha,2}(h_2 \circ \nabla v)(t) - \int_\Omega F(u,v)dx. \end{aligned} \tag{4.7}$$

Proof. Differentiating I and using equations in (P_3), integrating by parts, and using Young's inequality, (**A.5**), and Lemma 4.2, we get

$$\begin{aligned} I'(t) &= \|u_t\|_2^2 - \left(1 - \int_0^t g_1(s)ds\right)\|\nabla u\|_2^2 + \int_\Omega \nabla u(t) \cdot \int_0^t g_1(t-s)(\nabla u(s) - \nabla u(t))dsdx \\ &\quad + \|v_t\|_2^2 - \left(1 - \int_0^t g_2(s)ds\right)\|\nabla v\|_2^2 + \int_\Omega \nabla v(t) \cdot \int_0^t g_2(t-s)(\nabla v(s) - \nabla v(t))dsdx \\ &\quad - \int_\Omega [u f_1(u,v) + v f_2(u,v)]dx \\ &\leq \|u_t\|_2^2 - l_1\|\nabla u\|_2^2 + \frac{l_1}{2}\|\nabla u\|_2^2 + \frac{1}{2l_1}\int_\Omega \left(\int_0^t g_1(t-s)|\nabla u(s) - \nabla u(t)|ds\right)^2 dx \\ &\quad + \|v_t\|_2^2 - l_2\|\nabla v\|_2^2 + \frac{l_2}{2}\|\nabla u\|_2^2 + \frac{1}{2l_2}\int_\Omega \left(\int_0^t g_2(t-s)|\nabla v(s) - \nabla v(t)|ds\right)^2 dx \\ &\quad - \int_\Omega F(u,v)dx \\ &\leq \|u_t\|_2^2 - \frac{l_1}{2}\|\nabla u\|_2^2 + cC_{\alpha,1}(h_1 \circ \nabla u)(t) \\ &\quad + \|v_t\|_2^2 - \frac{l_2}{2}\|\nabla v\|_2^2 + cC_{\alpha,2}(h_2 \circ \nabla v)(t) - \int_\Omega F(u,v)dx. \end{aligned}$$

∎

Lemma 4.4 *Assume that* (**A.3**) *–* (**A.5**) *hold. Then, the functional* K *defined by*

$$K(t) := K_1(t) + K_2(t)$$

with

$$K_1(t) := -\int_\Omega u_t \int_0^t g_1(t-s)(u(t)-u(s))dsdx$$

and

$$K_2(t) := -\int_\Omega u_t \int_0^t g_2(t-s)(v(t)-v(s))dsdx$$

satisfies, along the solution of (P_3) *and for any* $0 < \delta < 1$*, the estimate*

$$\begin{aligned} K'(t) \quad \leq \quad & -\left(\int_0^t g_1(s)ds - \delta\right) \|u_t\|_2^2 + c\delta\|\nabla u\|_2^2 + \frac{c}{\delta}(C_{\alpha,1}+1)(h_1 \circ \nabla u)(t) \\ & -\left(\int_0^t g_2(s)ds - \delta\right) \|v_t\|_2^2 + c\delta\|\nabla v\|_2^2 + \frac{c}{\delta}(C_{\alpha,2}+1)(h_2 \circ \nabla v)(t). \quad (4.8) \end{aligned}$$

Proof. By exploiting equations in (P_3) and integrating by parts, we have

$$\begin{aligned} K_1'(t) \quad = \quad & \left(1 - \int_0^t g_1(s)ds\right) \int_\Omega \nabla u(t) \cdot \int_0^t g_1(t-s)(\nabla u(t) - \nabla u(s))dsdx \\ & + \int_\Omega \left(\int_0^t g_1(t-s)|\nabla u(t) - \nabla u(s)|ds\right)^2 dx \\ & + \int_\Omega f_1(u,v) \int_0^t g_1(t-s)(u(t)-u(s))dsdx \\ & - \int_\Omega u_t \int_0^t g_1'(t-s)(u(t)-u(s))dsdx - \left(\int_0^t g_1(s)ds\right) \|u_t\|_2^2. \end{aligned}$$

Now, we estimate the terms in the right-hand side of the above equality.

Applying Young's inequality and Lemma 4.2, we obtain, for any $0 < \delta < 1$,

$$
\begin{aligned}
\left(1 - \int_0^t g_1(s)ds\right) \int_\Omega \nabla u(t) \cdot \int_0^t g_1(t-s)(\nabla u(t) - \nabla u(s))dsdx \\
+ \int_\Omega \left(\int_0^t g_1(t-s)|\nabla u(t) - \nabla u(s)|ds\right)^2 dx \\
\leq \delta \|\nabla u\|_2^2 + \frac{c}{\delta} \int_\Omega \left(\int_0^t g_1(t-s)|\nabla u(t) - \nabla u(s)|ds\right)^2 dx \\
\leq \delta \|\nabla u\|_2^2 + \frac{c}{\delta} C_{\alpha,1}(h_1 \circ \nabla u)(t).
\end{aligned}
$$

Using $\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \leq 2E(t) \leq 2E(0)$ and inequalities (4.2) and (4.6), we have

$$
\begin{aligned}
&\int_\Omega f_1(u,v) \int_0^t g_1(t-s)(u(t) - u(s))dsdx \\
&\leq c\delta \int_\Omega \left(|u|^2 + |v|^2 + |u|^{2\beta_1} + |v|^{2\beta_2}\right) dx \\
&+ \frac{c}{\delta} \int_\Omega \left(\int_0^t g_1(t-s)(u(t) - u(s))ds\right)^2 dx \\
&\leq c\delta \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|\nabla u\|_2^{2\beta_1} + \|\nabla v\|_2^{2\beta_2}\right) \\
&+ \frac{c}{\delta} C_{\alpha,1}(h_1 \circ \nabla u)(t) \\
&= c\delta \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|\nabla u\|_2^{2(\beta_1 - 1)} \|\nabla u\|_2^2 + \|\nabla v\|_2^{2(\beta_2 - 1)} \|\nabla v\|_2^2\right) \\
&+ \frac{c}{\delta} C_{\alpha,1}(h_1 \circ \nabla u)(t) \\
&\leq c\delta \|\nabla u\|_2^2 + c\delta \|\nabla v\|_2^2 + \frac{c}{\delta} C_{\alpha,1}(h_1 \circ \nabla u)(t).
\end{aligned}
$$

Exploiting Young's inequality and Lemma 4.2 again, we obtain, for any $0 < \delta < 1$,

$$
\begin{aligned}
-\int_\Omega u_t \int_0^t g_1'(t-s)(u(t)-u(s))dsdx \\
&= \int_\Omega u_t \int_0^t h_1(t-s)(u(t)-u(s))dsdx \\
&- \int_\Omega u_t \int_0^t \alpha g_1(t-s)(u(t)-u(s))dsdx \\
&\leq \frac{\delta}{2}\|u_t\|_2^2 + \frac{1}{2\delta}\int_\Omega \left(\int_0^t \sqrt{h_1(t-s)}\sqrt{h_1(t-s)}(u(t)-u(s))ds\right)^2 dx \\
&+ \frac{\delta}{2}\|u_t\|_2^2 + \frac{1}{2\delta}\alpha^2 \int_\Omega \left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)^2 dx \\
&\leq \delta\|u_t\|_2^2 + \frac{1}{2\delta}\left(\int_0^t h_1(s)ds\right)(h_1 \circ \psi)(t) + \frac{c}{\delta}C_{\alpha,1}(h_1 \circ u)(t) \\
&\leq \delta\|u_t\|_2^2 + \frac{c}{\delta}(C_{\alpha,1}+1)(h_1 \circ \nabla u)(t).
\end{aligned}
$$

A combination of all the above estimates gives

$$K_1'(t) \leq -\left(\int_0^t g_1(s)ds - \delta\right)\|u_t\|_2^2 + c\delta\|\nabla u\|_2^2 + \frac{c}{\delta}(C_{\alpha,1}+1)(h_1 \circ \nabla u)(t) + c\delta\|\nabla v\|_2^2.$$

Similarly, we have

$$K_2'(t) \leq -\left(\int_0^t g_2(s)ds - \delta\right)\|v_t\|_2^2 + c\delta\|\nabla v\|_2^2 + \frac{c}{\delta}(C_{\alpha,2}+1)(h_2 \circ \nabla v)(t) + c\delta\|\nabla u\|_2^2.$$

The last two estimates lead to the desired result. ∎

Lemma 4.5 ([23]) *Assume that (**A.3**) – (**A.5**) hold. Then, the functionals J_1 and J_2 defined by*

$$J_1(t) := \int_\Omega \int_0^t G_1(t-s)|\nabla u(s)|^2 dsdx$$

and

$$J_2(t) := \int_\Omega \int_0^t G_2(t-s)|\nabla v(s)|^2 ds dx$$

with $G_i(t) := \displaystyle\int_t^\infty g_i(s)ds$ *(for* $i = 1, 2$*) satisfy, along the solution of* (P_3)*, the estimates*

$$J_1'(t) \leq 3(1-l)\|\nabla u\|_2^2 - \frac{1}{2}(g_1 \circ \nabla u)(t) \tag{4.9}$$

and

$$J_2'(t) \leq 3(1-l)\|\nabla v\|_2^2 - \frac{1}{2}(g_2 \circ \nabla v)(t), \tag{4.10}$$

where $l = \min\{l_1, l_2\}$*.*

Lemma 4.6 *The functional* $\mathcal{L}$ *defined by*

$$\mathcal{L}(t) := NE(t) + N_1 I(t) + N_2 K(t)$$

satisfies, for a suitable choice of N, N_1, $N_2 \geq 1$,

$$\mathcal{L}(t) \sim E(t) \tag{4.11}$$

and the estimate

$$\begin{aligned} \mathcal{L}'(t) \quad &\leq \quad -4(1-l)\big(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\big) - \big(\|u_t\|_2^2 + \|v_t\|_2^2\big) \\ &\quad -c\int_\Omega F(u,v)dx + \frac{1}{4}\big[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\big], \qquad \forall\, t \geq t_0, \end{aligned} \tag{4.12}$$

where $l = \min\{l_1, l_2\}$ *and* t_0 *has been introduced in Remark 4.1.*

Proof. It is not difficult to establish that $\mathcal{L}(t) \sim E(t)$. To prove (4.12), set

$$g_0 = \min\left\{\int_0^{t_0} g_1(s)ds, \int_0^{t_0} g_2(s)ds\right\} > 0, \quad \delta = \frac{l}{4cN_2} \quad \text{and} \quad C_\alpha = \max\{C_{\alpha,1},\, C_{\alpha,2}\}.$$

Exploiting (4.7), (4.8) and recalling that $g_i' = \alpha g_i - h_i$, we obtain, for any $t \geq t_0$,

$$\begin{aligned}
\mathcal{L}'(t) \quad \leq \quad & -\frac{l}{4}(2N_1 - 1)\big(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\big) - \left(g_0 N_2 - \frac{l}{4c} - N_1\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\
& -N_1 \int_\Omega F(u,v)dx + \frac{\alpha}{2}N\big[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\big] \\
& -\left[\frac{1}{2}N - \frac{4c^2}{l}N_2^2 - C_\alpha\left(\frac{4c^2}{l}N_2^2 + cN_1\right)\right]\big[(h_1 \circ \nabla u)(t) + (h_2 \circ \nabla v)(t)\big].
\end{aligned}$$

We start by choosing N_1 large enough so that

$$\frac{l}{4}(2N_1 - 1) > 4(1-l),$$

then we select N_2 so large that

$$g_0 N_2 - \frac{l}{4c} - N_1 > 1.$$

As $\dfrac{\alpha g_i^2(s)}{\alpha g_i(s) - g_i'(s)} < g_i(s)$ for $i = 1, 2$, it follows from the Lebesgue Dominated Convergence Theorem that

$$\lim_{\alpha \to 0^+} \alpha C_{\alpha,i} = \lim_{\alpha \to 0^+} \int_0^\infty \frac{\alpha g_i^2(s)}{\alpha g_i(s) - g_i'(s)} ds = 0 \qquad \text{for} \qquad i = 1, 2.$$

This gives

$$\lim_{\alpha \to 0^+} \alpha C_\alpha = 0.$$

Consequently, there exists $\alpha_0 \in (0,1)$ such that if $\alpha < \alpha_0$, then

$$\alpha C_\alpha < \frac{1}{8\left[\frac{4c^2}{l} N_2^2 + cN_1\right]}.$$

Now, we choose N large enough so that

$$N > \max\left\{\frac{16c^2}{l} N_2^2,\ \frac{1}{2\alpha_0}\right\}$$

and set

$$\alpha = \frac{1}{2N}.$$

Then

$$\frac{1}{4}N - \frac{4c^2}{l} N_2^2 > 0 \qquad \text{and} \qquad \alpha = \frac{1}{2N} < \alpha_0.$$

These imply

$$\begin{aligned} \frac{1}{2}N - \frac{4c^2}{l} N_2^2 - C_\alpha \left[\frac{4c^2}{l} N_2^2 + cN_1\right] \quad &> \quad \frac{1}{2}N - \frac{4c^2}{l} N_2^2 - \frac{1}{8\alpha} \\ &= \quad \frac{1}{4}N - \frac{4c^2}{l} N_2^2 > 0. \end{aligned}$$

Hence, we arrive at the required estimate. ∎

4.3 General Decay Result

In this section, we state and prove our main result.

Theorem 4.2 *Let* $(u_0, u_1), (v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *be given. Suppose that assumptions (**A.3**) –(**A.5**) hold. Then there exist two positive constants* k_1 *and* k_2 *such that the solution to problem* (P_3) *satisfies the estimate*

$$E(t) \leq k_2 G_*^{-1}\left(k_1 \int_{t_0}^{t} \xi(s)ds\right), \qquad \forall\, t > t_0, \tag{4.13}$$

where $t_0 = \min\{t_1, t_2\}$ *is introduced in Remark 4.1,* $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$ *and* G_* *is given by*

$$G_*(t) = \int_t^r \frac{1}{sG(s)} ds \qquad \text{with} \qquad G(t) = \min\{H_1'(t),\, H_2'(t)\}.$$

Proof. We start by using estimates (4.1) and (4.4) to deduce, for any $t \geq t_0$,

$$\begin{aligned}
\int_0^{t_0} g_1(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds + \int_0^{t_0} g_2(s)\|\nabla v(t) - \nabla v(t-s)\|_2^2 ds \\
\leq \frac{1}{\xi_1(t_0)} \int_0^{t_0} \xi_1(s) g_1(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds \\
+ \frac{1}{\xi_2(t_0)} \int_0^{t_0} \xi_2(s) g_2(s)\|\nabla v(t) - \nabla v(t-s)\|_2^2 ds \\
\leq -\frac{g_1(0)}{a_1\xi_1(t_0)} \int_0^{t_0} g_1'(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds \\
- \frac{g_2(0)}{a_2\xi_2(t_0)} \int_0^{t_0} g_2'(s)\|\nabla v(t) - \nabla v(t-s)\|_2^2 ds \\
\leq -cE'(t).
\end{aligned}$$

Exploiting this estimate, inequality (4.12) becomes, for some $m > 0$ and for any $t \geq t_0$,

$$\begin{aligned} \mathcal{L}'(t) &\leq -mE(t) + c\big[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\big] \\ &\leq -mE(t) - cE'(t) + c\int_{t_0}^{t} g_1(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds \\ &\quad +c\int_{t_0}^{t} g_2(s)\|\nabla v(t) - \nabla v(t-s)\|_2^2 ds. \end{aligned}$$

By setting $\mathcal{F} := \mathcal{L} + cE \sim E$, we obtain

$$\begin{aligned} \mathcal{F}'(t) &\leq -mE(t) + c\int_{t_0}^{t} g_1(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds \\ &\quad +c\int_{t_0}^{t} g_2(s)\|\nabla v(t) - \nabla v(t-s)\|_2^2 ds, \qquad \forall\, t \geq t_0. \end{aligned} \tag{4.14}$$

<u>Case I</u> H_1 and H_2 are linear: Set $\xi(t) = \min\{\xi_1(t), \xi_2(t)\} > 0$, for any $t \geq 0$, then ξ is differentiable almost everywhere and non-increasing on $[0, +\infty)$. Multiply both sides of (4.14) by $\xi(t)$ and exploit (**A.4**) and (4.4) to get

$$\begin{aligned} \xi(t)\mathcal{F}'(t) &\leq -m\xi(t)E(t) + c\xi(t)\int_{t_0}^{t} g_1(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds \\ &\quad +c\xi(t)\int_{t_0}^{t} g_2(s)\|\nabla v(t) - \nabla v(t-s)\|_2^2 ds \\ &\leq -m\xi(t)E(t) + c\int_{t_0}^{t} \xi_1(s)g_1(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds \\ &\quad +c\int_{t_0}^{t} \xi_2(s)g_2(s)\|\nabla v(t) - \nabla v(t-s)\|_2^2 ds \\ &\leq -m\xi(t)E(t) - c\int_{t_0}^{t} g_1'(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds \\ &\quad c\int_{t_0}^{t} g_2'(s)\|\nabla v(t) - \nabla v(t-s)\|_2^2 ds \\ &\leq -m\xi(t)E(t) - cE'(t), \qquad \forall\, t \geq t_0. \end{aligned}$$

Using the non-increasing property of ξ we have $\xi\mathcal{F} + cE \sim E$ and

$$(\xi\mathcal{F} + cE)'(t) \leq -m\xi(t)E(t), \qquad \forall\, t \geq t_0.$$

A simple integration over (t_0, t) yields, for two positive constants k_1 and k_2,

$$E(t) \leq k_2 \exp\left(-k_1 \int_{t_0}^{t} \xi(s)ds\right), \qquad \forall\, t > t_0.$$

Continuity of E (See [23]) gives

$$E(t) \leq k_2 \exp\left(-k_1 \int_{0}^{t} \xi(s)ds\right), \qquad \forall\, t > 0.$$

Case II H_1 or H_2 is nonlinear: First, we use Lemmas 4.5 and 4.6 to conclude that

$$\mathscr{L}(t) := \mathcal{L}(t) + J_1(t) + J_2(t)$$

is nonnegative and satisfies, for any $t \geq t_0$,

$$\begin{aligned}
\mathscr{L}'(t) \;\; &\leq \;\; -\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) - \left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\
&\quad\; - \int_\Omega F(u,v)dx - \frac{1}{4}\left[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right] \\
&\leq \;\; -\beta E(t),
\end{aligned}$$

for some $\beta > 0$. Consequently, we arrive at

$$\int_0^{\infty} E(s)ds < +\infty \tag{4.15}$$

and

$$E(t) \leq \frac{c}{t - t_0}, \qquad \forall\, t \geq t_0. \tag{4.16}$$

Now, we define functionals η_i (for $i = 1, 2$) by

$$\eta_1(t) := \gamma \int_{t_0}^{t} \|\nabla u(t) - \nabla u(t-s)\|_2^2 \, ds$$

and

$$\eta_2(t) := \gamma \int_{t_0}^{t} \|\nabla v(t) - \nabla v(t-s)\|_2^2 \, ds,$$

where (4.15) allows us to choose $0 < \gamma < 1$ so that

$$\eta_i(t) < 1, \qquad \forall\, t \geq t_0 \qquad \text{and} \qquad i = 1, 2. \tag{4.17}$$

We further assume that $\eta_i(t) > 0$, for any $t > t_0$. Also, we define another functional θ_i (for $i = 1, 2$) by

$$\theta_1(t) := -\int_{t_0}^{t} g_1'(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds,$$

$$\theta_2(t) := -\int_{t_0}^{t} g_2'(s)\|\nabla v(t) - \nabla v(t-s)\|_2^2 ds$$

and observe that

$$\theta_1(t) + \theta_2(t) \leq -cE'(t), \qquad \forall\, t \geq t_0. \tag{4.18}$$

In addition, it follows from the strict convexity of H_i and the fact that $H_i(0) = 0$ that

$$H_i(s\tau) \leq sH_i(\tau), \qquad \text{for} \qquad 0 \leq s \leq 1, \qquad \tau \in (0, r] \qquad \text{and} \qquad i = 1, 2.$$

These facts, hypothesis (**A.4)**, estimates (4.17) and Jensen's inequality lead to

$$\begin{aligned}
\theta_1(t) &= -\frac{1}{\gamma\eta_1(t)} \int_{t_0}^{t} \gamma\eta_1(t)g_1'(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds \\
&\geq \frac{1}{\gamma\eta_1(t)} \int_{t_0}^{t} \gamma\eta_1(t)\xi_1(s)H_1(g_1(s))\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds \\
&\geq \frac{\xi_1(t)}{\gamma\eta_1(t)} \int_{t_0}^{t} \gamma H_1(\eta_1(t)g_1(s))\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds \\
&\geq \frac{\xi_1(t)}{\gamma} H_1\left(\frac{1}{\eta_1(t)} \int_{t_0}^{t} \gamma\eta_1(t)g_1(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds\right) \\
&= \frac{\xi_1(t)}{\gamma} H_1\left(\gamma \int_{t_0}^{t} g_1(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds\right) \\
&= \frac{\xi_1(t)}{\gamma} \bar{H}_1\left(\gamma \int_{t_0}^{t} g_1(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds\right), \qquad \forall\, t \geq t_0,
\end{aligned}$$

where $\bar{H}_1$ is a C^2 extension of H_1 that is strictly increasing and strictly convex on $(0, \infty)$. This implies that

$$\int_{t_0}^{t} g_1(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds \leq \frac{1}{\gamma}\bar{H}_1^{-1}\left(\frac{\gamma\theta_1(t)}{\xi_1(t)}\right), \qquad \forall\, t \geq t_0.$$

Similarly, we have

$$\int_{t_0}^{t} g_2(s)\|\nabla v(t)-\nabla v(t-s)\|_2^2 ds \leq \frac{1}{\gamma}\bar{H}_2^{-1}\left(\frac{\gamma\theta_2(t)}{\xi_2(t)}\right), \qquad \forall\, t \geq t_0.$$

Thus, (4.14) becomes

$$\mathcal{F}'(t) \leq -mE(t) + c\bar{H}_1^{-1}\left(\frac{\gamma\theta_1(t)}{\xi_1(t)}\right) + c\bar{H}_2^{-1}\left(\frac{\gamma\theta_2(t)}{\xi_2(t)}\right), \qquad \forall\, t \geq t_0. \tag{4.19}$$

Set $G = \min\{\bar{H}_1', \bar{H}_2'\}$ and, for a fixed $0 < \varepsilon < r$, define a functional $\mathcal{F}_1$ by

$$\mathcal{F}_1(t) := G\left(\varepsilon\frac{E(t)}{E(0)}\right)\mathcal{F}(t) + E(t), \qquad \forall\, t \geq 0.$$

Then, using the fact that $E' \leq 0$, $\bar{H}_i' > 0$ and $\bar{H}_i'' > 0$, we deduce that $\mathcal{F}_1 \sim E$ and, we, further, have,

$$\mathcal{F}_1'(t) = \varepsilon\frac{E'(t)}{E(0)}G'\left(\varepsilon_1\frac{E(t)}{E(0)}\right)\mathcal{F}(t) + G\left(\varepsilon\frac{E(t)}{E(0)}\right)\mathcal{F}'(t) + E'(t), \qquad \text{for} \quad a.e \;\; t \geq t_0.$$

By dropping the first and last terms of the above identity, since they are non-positive, and using estimate (4.19), we get

$$\begin{aligned}\mathcal{F}_1'(t) \;\; \leq \;\; & -mE(t)G\left(\varepsilon\frac{E(t)}{E(0)}\right) + cG\left(\varepsilon\frac{E(t)}{E(0)}\right)\bar{H}_1^{-1}\left(\frac{\gamma\theta_1(t)}{\xi_1(t)}\right) \\ & + cG\left(\varepsilon\frac{E(t)}{E(0)}\right)\bar{H}_2^{-1}\left(\frac{\gamma\theta_2(t)}{\xi_2(t)}\right), \qquad \text{for} \quad a.e \;\; t \geq t_0.\end{aligned} \tag{4.20}$$

Let $\bar{H}_i^*$ be the convex conjugate of $\bar{H}_i$ in the sense of Young (see [80, pp. 61-64]), which has the form

$$\bar{H}_i^*(s) = s\big(\bar{H}_i'\big)^{-1}(s) - \bar{H}_i\left[\big(\bar{H}_i'\big)^{-1}(s)\right], \qquad \text{for} \qquad i = 1, 2, \tag{4.21}$$

and satisfies the following generalized Young's inequality

$$AB_i \le \bar{H}_i^*(A) + \bar{H}_i(B_i), \qquad \text{for} \qquad i = 1, 2. \tag{4.22}$$

By taking $A = G\left(\varepsilon\dfrac{E(t)}{E(0)}\right)$, $B_i = \bar{H}_i^{-1}\left(\dfrac{\gamma\theta_i(t)}{\xi_i(t)}\right)$, for $i = 1, 2$, and combining (4.20) – (4.22), we obtain, for almost every $t \ge t_0$,

$$\begin{aligned}
\mathcal{F}_1'(t) &\le -mE(t)G\left(\varepsilon\frac{E(t)}{E(0)}\right) + c\bar{H}_1^*\left[G\left(\varepsilon\frac{E(t)}{E(0)}\right)\right] + c\frac{\gamma\theta_1(t)}{\xi_1(t)} \\
&\quad + c\bar{H}_2^*\left[G\left(\varepsilon\frac{E(t)}{E(0)}\right)\right] + c\frac{\gamma\theta_2(t)}{\xi_2(t)} \\
&\le -mE(t)G\left(\varepsilon\frac{E(t)}{E(0)}\right) + cG\left(\varepsilon\frac{E(t)}{E(0)}\right)\big(\bar{H}_1'\big)^{-1}\left[G\left(\varepsilon\frac{E(t)}{E(0)}\right)\right] + c\frac{\gamma\theta_1(t)}{\xi_1(t)} \\
&\quad + cG\left(\varepsilon\frac{E(t)}{E(0)}\right)\big(\bar{H}_2'\big)^{-1}\left[G\left(\varepsilon\frac{E(t)}{E(0)}\right)\right] + c\frac{\gamma\theta_2(t)}{\xi_2(t)} \\
&\le -mE(t)G\left(\varepsilon\frac{E(t)}{E(0)}\right) + cG\left(\varepsilon\frac{E(t)}{E(0)}\right)\big(\bar{H}_1'\big)^{-1}\left[\bar{H}_1'\left(\varepsilon\frac{E(t)}{E(0)}\right)\right] + c\frac{\gamma\theta_1(t)}{\xi_1(t)} \\
&\quad + cG\left(\varepsilon\frac{E(t)}{E(0)}\right)\big(\bar{H}_2'\big)^{-1}\left[\bar{H}_2'\left(\varepsilon\frac{E(t)}{E(0)}\right)\right] + c\frac{\gamma\theta_2(t)}{\xi_2(t)} \\
&\le -\big(mE(0) - c\varepsilon\big)\frac{E(t)}{E(0)}G\left(\varepsilon\frac{E(t)}{E(0)}\right) + c\left(\frac{\gamma\theta_1(t)}{\xi_1(t)} + \frac{\gamma\theta_2(t)}{\xi_2(t)}\right).
\end{aligned}$$

Multiplying this estimate by $\xi(t) = \min\{\xi_1(t), \xi_2(t)\} > 0$ and using inequality (4.18), we obtain

$$\xi(t)\mathcal{F}_1'(t) \le -(mE(0) - c\varepsilon)\xi(t)\frac{E(t)}{E(0)}G\left(\varepsilon\frac{E(t)}{E(0)}\right) + c\gamma\big(\theta_1(t) + \theta_2(t)\big)$$

$$\leq \quad -(mE(0)-c\varepsilon)\xi(t)\frac{E(t)}{E(0)}G\left(\varepsilon\frac{E(t)}{E(0)}\right)-cE'(t), \qquad \text{for} \quad a.e \ \ t\geq t_0.$$

Take ε smaller, if needed, to get, for some $k_0>0$,

$$\xi(t)\mathcal{F}_1'(t)\leq -k_0\xi(t)\frac{E(t)}{E(0)}G\left(\varepsilon\frac{E(t)}{E(0)}\right)-cE'(t), \qquad \text{for} \quad a.e \ \ t\geq t_0.$$

Consequently, by setting $\mathcal{F}_2=\xi\mathcal{F}_1+cE$, we obtain, for some $\alpha_1,\alpha_2>0$

$$\alpha_1\mathcal{F}_2(t)\leq E(t)\leq \alpha_2\mathcal{F}_2(t), \qquad \forall\, t\geq t_0 \tag{4.23}$$

and

$$\mathcal{F}_2'(t)\leq -k_0\xi(t)\frac{E(t)}{E(0)}G\left(\varepsilon\frac{E(t)}{E(0)}\right), \qquad \text{for} \quad a.e \ \ t\geq t_0. \tag{4.24}$$

It follows from $0\leq \varepsilon\dfrac{E(t)}{E(0)}<r$ that

$$\begin{aligned} G\left(\varepsilon\frac{E(t)}{E(0)}\right) &= \min\left\{\bar{H}_1'\left(\varepsilon\frac{E(t)}{E(0)}\right),\bar{H}_2'\left(\varepsilon\frac{E(t)}{E(0)}\right)\right\} \\ &= \min\left\{H_1'\left(\varepsilon\frac{E(t)}{E(0)}\right),H_2'\left(\varepsilon\frac{E(t)}{E(0)}\right)\right\}, \qquad \forall\, t\geq 0. \end{aligned}$$

Now, set

$$G_0(\tau)=\tau G(\varepsilon\tau), \qquad \forall\, \tau\in[0,1],$$

we deduce from $H_i'>0$ and $H_i''>0$ on $(0,r]$ (for $i=1,2$), that, $G_0,G_0'>0$ $a.e.$ on $(0,1]$. Define a functional R by

$$R(t):=\frac{\alpha_1\mathcal{F}_2(t)}{E(0)}$$

and exploit (4.23) and (4.24) to notice that $R \sim E$ and, for some $k_1 > 0$,

$$R'(t) \leq -k_1 \xi(t) G_0(R(t)), \qquad \text{for} \quad a.e \;\; t \geq t_0.$$

An integration over (t_0, t) gives

$$-\int_{t_0}^{t} \frac{R'(s)}{G_0(R(s))} ds \geq k_1 \int_{t_0}^{t} \xi(s) ds$$

or equivalently,

$$\int_{\varepsilon R(t)}^{\varepsilon R(t_0)} \frac{1}{sG(s)} ds \geq k_1 \int_{t_0}^{t} \xi(s) ds;$$

which implies that

$$R(t) \leq \frac{1}{\varepsilon} G_*^{-1} \left(k_1 \int_{t_0}^{t} \xi(s) ds \right) \qquad \forall\, t > t_0,$$

where $G_*(t) := \displaystyle\int_t^r \frac{1}{sG(s)} ds$. A combination of this estimate with the fact that $R \sim E$ gives

$$E(t) \leq k_2 G_*^{-1} \left(k_1 \int_{t_0}^{t} \xi(s) ds \right) \qquad \forall\, t > t_0.$$

- If $\eta_i(t) = 0$, for $t \geq t_0$ and $i = 1, 2$, then we get an exponential decay from (4.14).

- If $\eta_1(t) > 0$ and $\eta_2(t) = 0$, for any $t > t_0$, then we set $\theta_2(t) = 0$ and (4.19) becomes

$$\mathcal{F}'(t) \leq -mE(t) + c\bar{H}_1^{-1} \left(\frac{\gamma \theta_1(t)}{\xi_1(t)} \right), \qquad \forall\, t \geq t_0.$$

Repeating the above steps, we arrive at

$$E(t) \leq k_2 G_*^{-1}\left(k_1 \int_{t_0}^{t} \xi_1(s)ds\right) \qquad \forall\, t > t_0,$$

where

$$G_*(t) := \int_t^r \frac{1}{sH_1'(s)} ds.$$

- Similarly, if $\eta_1(t) = 0$ and $\eta_2(t) > 0$, for any $t > t_0$, we have

$$E(t) \leq k_2 G_*^{-1}\left(k_1 \int_{t_0}^{t} \xi_2(s)ds\right) \qquad \forall\, t > t_0,$$

where

$$G_*(t) := \int_t^r \frac{1}{sH_2'(s)} ds.$$

This completes the proof. ∎

Example 4.1

(1) Consider the relaxation functions $g_1(t) = ae^{-t}$ *and* $g_2(t) = \dfrac{b}{(1+t)^{\mu}}$, $\mu > 1$, *where* a *and* b *are chosen so that condition (***A.3***) is satisfied. Then there exists* $C > 0$ *such that*

$$E(t) \leq \frac{C}{(1+t)^{\mu}}, \qquad \forall\, t > t_0.$$

(2) Let $g_1(t) = \dfrac{a}{(1+t)^{\mu}}$ *and* $g_2(t) = \dfrac{b}{(1+t)^{\nu}}$ *with* $\mu, \nu > 1$, *where* a *and* b *are chosen so that condition (***A.3***) is satisfied. Then, there exists* $C > 0$ *such that,*

for any $t > t_0$,

$$E(t) \leq \frac{C}{(1+t)^{\gamma}}, \quad \text{with} \quad \gamma = \min\{\mu, \nu\}.$$

(3) If $g_1(t) = ae^{-t}$ *and* $g_2(t) = be^{-(1+t)^{\nu}}$ *with* $0 < \nu < 1$, *where* a *and* b *are chosen so that condition (**A.3**) is satisfied. Then, there exist positive constants* C *and* k_1 *such that*

$$E(t) \leq Ce^{-k_1(1+t)^{\nu}}, \qquad \text{for} \quad t \quad \text{large}.$$

(4) If $g_1(t) = ae^{-(1+t)^{\nu}}$ *with* $0 < \nu < 1$ *and* $g_2(t) = \dfrac{b}{(1+t)^{\mu}}$ *with* $\mu > 1$, *where* a *and* b *are chosen so that condition (**A.3**) is satisfied. Then, there exists* $C > 0$ *such that*

$$E(t) \leq \frac{C}{(1+t)^{\mu}}, \qquad \text{for} \quad t \quad \text{large}.$$

CHAPTER 5

CONCLUSIONS AND FUTURE WORKS

5.1 Conclusions

In this dissertation we studied the general decay for viscoelastic-type Timoshenko system, viscoelastic-type Bresse system and a coupled system of viscoelastic wave equations. We proved some energy decay results for Timoshenko and Bresse systems in the case of equal and non-equal speeds of wave propagation under the following condition on the relaxation function, for some $1 \leq p < \frac{3}{2}$,

$$g'(t) \leq -\xi(t) g^p(t), \qquad \forall\, t \geq 0.$$

Our results are the extension of that of Messaoudi and Al-Khulaifi [22] to the case of Timoshenko and Bresse systems and that of Mustafa [23] to the case of system of viscoelastic wave equations.

For Timoshenko system, in the case of equal speeds of wave propagation, our results are the generalization of many earlier results in the literature, [33], [34], [35].

For Bresse system, to the best of our knowledge, our results are the first to deal with the energy decay rate for a viscoelastic bresse system with finite memory. These results allow a wider class of relaxation functions.

We proved a new general decay rate result for a coupled system of viscoelastic wave equations with the following wider classes of relaxation functions, for $i = 1, 2$,

$$g_i'(t) \leq -\xi_i(t) H_i(g(t)), \qquad \forall\, t \geq 0.$$

Our result generalizes the ones in [68], [69], [70], [73].

5.2 Future Works

Investigating Viscoelastic-type Bresse and Timoshenko Systems with Robin Boundary Conditions

In Chapter 2, we studied a viscoelastic-type Timoshenko system with Dirichlet boundary conditions and proved general decay results for the system in the case of equal speeds of wave propagation as well as non-equal speed case. Similar results had been established in Chapter 3 for a viscoelastic-type Bresse system with Dirichlet-Neumann-Neumenn boundary conditions. These problems can be studied with Robin (mixed) boundary conditions and some general decay results can be established in the cases of equal and non-equal speeds of wave propagation.

Investigating System of two Viscoelastic Wave Equations with Nonlinear Damping Terms

Said-Houari *et al*. [71] considered the following system of viscoelastic wave equations with nonlinear damping terms acting on both equations

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g_1(t-s)\Delta u(s)ds + |u_t|^{m-2}u_t = f_1(u,v), & \text{in } \ \Omega \times (0,\infty), \\ v_{tt} - \Delta v + \displaystyle\int_0^t g_2(t-s)\Delta v(s)ds + |v_t|^{r-2}v_t = f_2(u,v), & \text{in } \ \Omega \times (0,\infty), \\ u = v = 0, & \text{on } \ \partial\Omega \times [0,\infty), \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, \quad v(\cdot,0) = v_0, \quad v_t(\cdot,0) = v_1, & \text{in } \ \Omega, \end{cases} \tag{5.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, u_0, v_0, v_0, v_1 are initial data,

$$f_1(u,v) = a|u+v|^{2(\rho+1)}(u+v) + b|u|^{\rho}u|v|^{\rho+2}$$

$$f_2(u,v) = a|u+v|^{2(\rho+1)}(u+v) + b|u|^{\rho+2}|v|^{\rho}v,$$

$$2 \leq m, r \quad \text{if } \ N = 1,2 \quad \mathring{and} \quad 2 \leq m, r \leq \frac{2N}{N-2} \quad \text{if } \ N \geq 3,$$

$$-1 \leq \rho \quad \text{if } \ N = 1,2 \quad \mathring{and} \quad -1 \leq \rho \leq \frac{3-N}{N-2} \quad \text{if } \ N \geq 3.$$

They established a general decay result for the solution of Problem (1.32) with the relaxation functions satisfying

$$g_i'(t) \leq \xi_i(t) g_i(t), \qquad \forall\, t \geq 0.$$

This result can be extended to the case where the relaxation functions satisfy assumptions (**A.4**).

Investigating a Weakly Dissipative Second Order Abstract Systems with Memory

Let H be a real separable Hilbert space whose associated inner product and norm are respectively denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$.

The modeling of the dynamics of physical phenomena such as heat flow in conductors with memory, hereditary polarization in dielectrics, population dynamics, viscolasticity can be described by an abstract integro-differential equation of the form

$$\begin{cases} u_{tt} + Au - \displaystyle\int_{-\infty}^{t} g(t-s) A^{\alpha} u(s) ds = 0, \quad t > 0, \\ u(-t) = u_0(t), \quad t \geq 0, \qquad u_t(0) = u_1, \end{cases} \tag{5.2}$$

where $A : \mathcal{D}(A) \subset H \longrightarrow H$ is a positive definite self-adjoint operator on H, g is the relaxation (convolution kernel) function, $\alpha \in [0,1]$, u_0, u_1 are given history function and initial data respectively.

Motivated by the works of Dafermos [3], [6], many results dealing with the existence, uniqueness, regularity and asymptotic behavior of many systems of the form (5.2) have been studied; see, for example, [81], [82], [83], [84]. In the case of finite history, that is, $u_0(t) = 0$ for $t < 0$, see [12], [13], [14], [15], [22], [23], [85]. In particular, Rivera *et al.* [85] considered the interpolating cases $\alpha \in (0,1)$ and a relaxation function g which decays exponentially to zero at infinity, that is,

$$-c_0 g(s) \leq g'(s) \leq -c_1 g(s) \qquad \forall\, s \in \mathbb{R}_+. \tag{5.3}$$

They showed that the energy decays polynomially at the rate of $\frac{1}{t}$.

We can study the class of viscoelastic equations of the form

$$\begin{cases} u_{tt} + Au(t) - \displaystyle\int_0^t g(t-s)A^{\alpha}u(s)ds = 0, \quad t > 0, \\ u(0) = u_0, \qquad u_t(0) = u_1, \end{cases} \tag{5.4}$$

where $A : \mathcal{D}(A) \subset H \longrightarrow H$ is a positive definite self-adjoint operator on H such that the embedding $\mathcal{D}(A^{\beta}) \hookrightarrow \mathcal{D}(A^{\sigma})$ is compact for any $\beta > \sigma \geq 0$ and $\alpha \in (0,1)$. The assumption $\mathcal{D}(A^{\beta}) \hookrightarrow\hookrightarrow \mathcal{D}(A^{\sigma})$ for any $\beta > \sigma \geq 0$ guarantees the existence of some positive constants $\omega_0 > 0$ such that

$$\left\|A^{\alpha/2}v\right\|^2 \leq \omega_0 \left\|A^{1/2}v\right\|^2 \qquad \forall\, v \in \mathcal{D}\left(A^{1/2}\right), \tag{5.5}$$

By imposing the following conditions on the relaxation function:

- $g : [0,\infty) \longrightarrow [0,\infty)$ is a non-increasing differentiable function satisfying

$$g(0) > 0 \qquad \text{and} \qquad 1 - \omega_0 \int_0^t g(s)ds = l > 0;$$

- there exist a nonincreasing function $\xi : [0,\infty) \longrightarrow (0,\infty)$ and a strictly convex function $G : [0, \infty) \longrightarrow [0, \infty)$ such that

$$g'(t) \leq -\xi(t)G(g(t)), \qquad \forall\, t \geq 0;$$

we will try to generalize and improve the result of Rivera *et al*. [85].

Investigating the Existence of Asymptotically Almost Periodic Solutions For Some Hyperbolic Integrodifferential Equations

Integrodifferential equations play an important role when it comes to describing various practical problems, see, e.g., [86], [87], [88], [89], [90], [91], [92], [93]. One often makes use of these types of differential equations to study practical problems in which some memory effect in taken into account. Among other things, integrodifferential equations of Gurtin-Pipkin type have been widely used to study various practical problems including the heat conduction in materials with memory or the sound propagation in viscoelastic media or in homogenization problems in perforated media (Darcy's Law), see, e.g., [86], [94], [95], [96], [97].

Let $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H)$ be a separable Hilbert space. We will investigate the existence of asymptotically almost periodic mild solutions to the class of hyperbolic integrodifferential equations of Gurtin-Pipkin type given by

$$\frac{d^2u}{dt^2} + A^2u - \int_{-\infty}^{t} g(t-s)A^2u(s)ds = f(t,u), \quad t > 0 \tag{5.6}$$

with initial conditions

$$u(-t) = u_0(t), \quad t \geq 0 \quad \text{and} \quad u'(0) = u_1, \tag{5.7}$$

where $A : \mathcal{D}(A) \subset H \mapsto H$ is a positive self-adjoint operator which is bounded below, that is, there exists a constant $\omega > 0$ such that

$$\|Au\|_H \geq \omega \|u\|_H \quad \text{for all} \quad u \in \mathcal{D}(A), \tag{5.8}$$

the function $f : [0, \infty) \times H \mapsto H$ is asymptotically almost periodic in the first variable uniformly in the second one, and the non-increasing differentiable relaxation (kernel) function $g : [0, \infty) \longrightarrow [0, \infty)$ satisfies the following assumptions,

(**A.1**) $g(0) > 0$ and $1 - \displaystyle\int_0^{+\infty} g(s)ds = \beta > 0$; and

(**A.2**) there exists a positive constant ξ such that $g'(t) \leq -\xi g(t)$ for all $t \geq 0$.

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- (1) Salim A. Messaoudi and Jamilu Hashim Hassan. *General and optimal decay in memory-type Timoshenko system*, Journal of Integral Equation and Applications Volume 30, Number 1 (2018), 117–45, DOI:10.1216/JIE-2018-30-1-117.

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